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Connection formulae for the fourth Painlevé transcendent; Clarkson–McLeod solution

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Abstract. Using the isomonodromy and Riemann–Hilbert methods, we perform a rigorous global asymptotic analysis of the Clarkson–McLeod fourth Painlevé transcendent. In particular, we prove some of the Clarkson–McLeod conjectures concerning the asymptotic behaviour, as $x \to -\infty$, of the solutions of Painlevé IV equation which decay as $x \to +\infty$. The relevant exact connection formulae are also rigorously derived.

1. Introduction

In this paper we consider solutions of the fourth Painlevé (PIV) equation,

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4xw^2 + (-4\alpha + \beta + 2x^2)w - \frac{\beta^2}{2w} \tag{1.1}$$

satisfying (under the assumption, $\beta = 0$) the boundary condition,

$$w(x) \to 0 \qquad \text{as } x \to +\infty. \tag{1.2}$$

The study of this class of Painlevé transcendents was initiated in the work of Clarkson and McLeod [1] and has been continued by Bassom, Clarkson, Hicks and McLeod (BCHM) in the subsequent series of papers (see [1–4]). In particular, in [3] it is proven that, in the case $\beta = 0$ and real α, any real solution of (1.1) satisfying boundary condition (1.2) is multiple of the square of the parabolic cylinder function $D_{\alpha-\frac{1}{2}}(\sqrt{2}x)$, i.e.

$$w(x) \sim k^2 2^{3/2} D^2_{\alpha-\frac{1}{2}}(\sqrt{2}x) \sim k^2 2^{\alpha+1} x^{2\alpha-1} \mathrm{e}^{-x^2} \tag{1.3}$$

for some constant $k^2 \in \mathbb{R}$. Moreover, as it is also proven in [3], for any k^2, there exists a unique solution of (1.1) ($\beta = 0$) asymptotic to $k^2 2^{3/2} D^2_{\alpha-\frac{1}{2}}(\sqrt{2}x)$.

The one-parameter family $w(x; k^2)$ of solutions of the PIV equation (1.1) determined by asymptotic condition (1.3) was first introduced in [1]. We will refer to the function $w(x; k^2)$ as to *Clarkson–McLeod PIV transcendent*.

One of the principal questions related to $w(x; k^2)$ is its behaviour as $x \to -\infty$. In [1] it is conjectured that there exists the value $k^* \equiv k^*(\alpha) > 0$ such that:

(1) if $|k| < k^*$ then $w(x)$ is smooth for all real x, and as $x \to -\infty$,

$$w(x) \sim \kappa_n^2 2^{\alpha+1} x^{2\alpha-1} \mathrm{e}^{-x^2} = \kappa_n^2 2^{n+\frac{3}{2}} x^{2n} \mathrm{e}^{-x^2} \qquad \alpha - \tfrac{1}{2} = n \in \mathbb{N} \tag{1.4}$$

or

$$w(x) = -\frac{2x}{3} + (-1)^{[\alpha+\frac{1}{2}]}\frac{4d}{\sqrt{3}}\sin\left(\frac{x^2}{\sqrt{3}} - \frac{4d^2}{\sqrt{3}}\ln(-\sqrt{2}x) + c + \mathcal{O}(x^{-2})\right) + \mathcal{O}\left(\frac{1}{x}\right) \qquad \alpha - \frac{1}{2} \notin \mathbb{Z} \tag{1.5}$$

where $[\alpha + \frac{1}{2}]$ denotes the integer part of $\alpha + \frac{1}{2}$ and the constants κ_n, d, c are dependent on k;

(2) if $|k| = k^*$ then as $x \to -\infty$

$$w(x) \sim -2x \tag{1.6}$$

(3) if $|k| > k^*$ then $w(x)$ has a pole at some point $x_0 \equiv x_0(k) \in \mathbb{R}$.

In the case $\alpha - \frac{1}{2} = n \in \mathbb{N}$, part (1) of this conjecture was proven in [1, 3] and the following equations for the values of κ_n and k^* were obtained:

$$\begin{aligned} \kappa_n^2(k) &= \frac{k^2}{1 - 2\sqrt{2\pi}n!k^2} \\ (k^*)^2 &= \frac{1}{2\sqrt{2\pi}n!}. \end{aligned} \tag{1.7}$$

In [1], it was also suggested that the following generalization of equation (1.7) for non-integer values of $\alpha - \frac{1}{2}$ takes place:

$$(k^*)^2 = \frac{1}{2\sqrt{2\pi}\,\Gamma(\alpha + \frac{1}{2})} \qquad \left(\alpha + \frac{1}{2} > 0\right). \tag{1.8}$$

(For some technical reason we use slightly different parametrization of the PIV equation than the one used in [1–4]. If α^{BCHM} and β^{BCHM} denote α, β-parameters in [3] then their relation to our α, β is given by equations

$$\alpha^{\mathrm{BCHM}} = 2\alpha - \frac{\beta}{2} \qquad \beta^{\mathrm{BCHM}} = -\frac{\beta^2}{2}.$$

Equations (1.5) and (1.8) were thoroughly investigated and verified numerically in [4, 2]. The problem of their rigorous justification and the problem of evaluation of the exact connection formulae for the asymptotic parameters $d(k)$ and $c(k)$ were left open. In this paper we address these two problems via the framework of the isomonodromy method (IM) (see [5, 6]; see also [7])

The first results concerning the application of the IM to the global asymptotic analysis of the PIV equation are due to Kitaev [8]. In [8] a complete description (including all the relevant connection formulae) of the asytmptotic behaviour of the general solution of equation (1.1) as $x \to \mathrm{e}^{\mathrm{i}\pi/4+\mathrm{i}\pi j}\infty$, $j \in \mathbb{Z}$ was obtained.

In this paper, we follow the general methodology of [9], i.e. we combine the IM, the Deift–Zhou nonlinear steepest-descent method [10], and the Kitaev method [11] for the justification of the asymptotic results obtained via the IM.

Our main result, which completes the proof (up to the error term) of part (1) of the Clarkson–McLeod conjecture and supplements it by exact connection formulae for the asymptotic parameters $d(k)$ and $c(k)$, can be formulated as in the following theorem.

Theorem 1.1. Let β, α, and k^2 be the real numbers such that,

$$\beta = 0$$
$$\alpha - \tfrac{1}{2} \notin \mathbb{Z}$$

and

$$0 < k^2 2\sqrt{2\pi}\,\Gamma\left(\alpha + \frac{1}{2}\right) \equiv \frac{k^2}{(k^*)^2} < 1 \tag{1.9}$$

and let $w(x; k^2)$ be the corresponding Clarkson–McLeod PIV transcendent, i.e. the unique solution of equation (1.1) satisfying the boundary condition (1.3) as $x \to +\infty$. Then, the Painlevé transcendent $w(x; k^2)$ is a meromorphic function of x whose asymptotic behaviour as $x \to -\infty$ is described by the equation

$$w(x) = -\frac{2x}{3} + 2\sqrt{2}a\cos\left(\frac{x^2}{\sqrt{3}} - \sqrt{3}a^2\ln(2\sqrt{3}\,x^2) + \phi\right) + \mathcal{O}((-x)^{-1/4}\ln(-x)) \tag{1.10}$$

where

$$\begin{aligned} a^2 &= -\frac{1}{2\sqrt{3}\pi}\ln(1 - |s_-|^2) \qquad a > 0 \\ \phi &= -\frac{3\pi}{4} - \frac{2\pi}{3}\alpha - \arg\Gamma(-\mathrm{i}\sqrt{3}a^2) - \arg s_- \\ s_- &= \text{constant} \end{aligned} \tag{1.11}$$

and the connection between the asymptotic coefficients k and s_- is given by

$$s_- = 1 - \frac{2(2\pi)^{3/2}\mathrm{e}^{-\mathrm{i}\pi\alpha}}{\Gamma(\frac{1}{2} - \alpha)}k^2. \tag{1.12}$$

We note that equation (1.10) coincides, up to the error term, with equation (1.5) and yields the following exact connection formulae for the parameters $d(k)$ and $c(k)$,

$$\begin{aligned} d^2 &= -\frac{\sqrt{3}}{4\pi}\ln(1 - |s_-|^2) \qquad d > 0 \\ c &= -\frac{\pi}{4} + \pi\left[\alpha + \frac{1}{2}\right] - \frac{2\pi}{3}\alpha - \frac{d^2}{\sqrt{3}}\ln 3 - \arg\Gamma\left(-\mathrm{i}\frac{2}{\sqrt{3}}d^2\right) - \arg s_- \end{aligned}$$

where

$$s_- \equiv s_-(k) = 1 - \frac{2(2\pi)^{3/2}\mathrm{e}^{-\mathrm{i}\pi\alpha}}{\Gamma(\frac{1}{2} - \alpha)}k^2.$$

Remark 1.1. The meromorphicity of $w(x; k^2)$ does not need to be proven; this is a well known classical fact concerning Painlevé transcendents. Its elegant modern proof based on the analysis of the corresponding Riemann–Hilbert problem is given in [12].

Remark 1.2. It should be emphasized that we do not claim that $w(x; k^2)$ does not have singularities on the real axes. Moreover, as it follows from the numerical analysis performed in [4, 2], $w(x; k^2)$ might blow up at finite x if $\alpha < -\frac{1}{2}$. At the same time, the same numerical results allows one to expect the absence of the real poles of $w(x; k^2;)$ if $\alpha > -\frac{1}{2}$.

Remark 1.3. As a by-product of the proof of theorem 1.1 (see remark 5.1 below) we also obtain the following *local* asymptotic result concerning the behaviour of the solutions of the PIV equation as $x \to -\infty$.

Theorem 1.2. Let $\beta, \alpha, a,$ and ϕ be the real numbers such that,

$$\alpha - \tfrac{1}{2} \notin \mathbb{Z}$$

and

$$a > 0$$

(note that we do not assume that $\beta = 0$). Then there exists a solution $w(x)$ of equation (1.1) which has the asymptotics indicated in (1.10) as $x \to -\infty$. This solution is unique if the error term, $\mathcal{O}((-x)^{-1/4}\ln(-x))$, in (1.10) can be replaced by $\mathcal{O}(x^{-1})$.

As it has already been mentioned, this is a local statement, which does not reflect the integrability of equation (1.1). In fact, in the case $\beta = 0$, $\alpha \in \mathbb{R}$ the existence of the two-parameter family $w(x; a, \phi)$ of solutions of (1.1) characterized by the asymptotics (1.10) with the error term $\mathcal{O}(x^{-1})$ has recently been proven by Abdulaev [13] without any use of the isomonodromy method, i.e. without any use of the integrability of the PIV equation. Moreover, combining the results of [13] with theorem 3.1 and corollary 4.1 below one can replace the error term in (1.10) by $\mathcal{O}(x^{-1})$ for any k^2 satisfying (1.9). This improvement can also be achieved using only the isomonodromy technique. However, that comes at the expense of much longer calculations.

Remark 1.4. Similar to the case of the second Painlevé equation (see [14]), the IM allows us to obtain a complete list of all possible asymptotics of the solutions of (1.1) as $x \to \infty$, $x \in \mathbb{C}$. We shall publish this list elsewhere.

2. Monodromy parametrization of the PIV transcendent. The Riemann–Hilbert problem

This section plays an important yet auxiliary role. For the reader's convenience we collect here, following [8], the necessary facts concerning the isomonodromy formalism for the PIV equation. The detailed proofs of the results presented in this section can be found in [8] and also in [15]. For the basic definitions and concepts related to the general monodromy theory of systems of ordinary differential equations (ODEs) with rational coefficients we refer the reader to the monograph [16] (see also [6]).

We shall use the Lax pair for equation (1.1) given in [8]:

$$\frac{\partial\Psi}{\partial\xi} = \left\{\left(\frac{1}{2}\xi^3 + \xi(x+uv) + \frac{\alpha}{\xi}\right)\sigma_3 + \mathrm{i}(\xi^2 u + 2xu + u')\sigma_+ + \mathrm{i}(\xi^2 v + 2xv - v')\sigma_-\right\}\Psi \tag{2.1}$$

$$\frac{\partial\Psi}{\partial x} = \left\{\left(\frac{1}{2}\xi^2 + uv\right)\sigma_3 + \mathrm{i}\xi u\sigma_+ + \mathrm{i}\xi v\sigma_-\right\}\Psi. \tag{2.2}$$

Here, $\Psi(\xi, x)$ is a 2×2 matrix function, and $\sigma_3, \sigma_+, \sigma_-$ denote the Pauli matrices:

$$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad \sigma_+ = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}.$$

The compatibility condition of equations (2.1) and (2.2) is equivalent to the following system of nonlinear ODEs:

$$\begin{cases} \alpha' = 0 \\ u'' + 2xu' + u + 2\alpha u - 4xu^2 v - 2vuu' = 0 \\ v'' - 2xv' - v + 2\alpha v - 4xuv^2 + 2uvv' = 0 \end{cases} \tag{2.3}$$

which in turn implies that

$$\beta \equiv u'v - uv' + 2xuv - (uv)^2 = \text{constant} \tag{2.4}$$

and the product

$$w = uv \tag{2.5}$$

satisfies equation (1.1). This means that equation (2.1) is the linear matrix ODE with rational coefficients whose monodromy data, according to the IM formalism (cf [5, 6]) form a complete set of the first integrals of Painlevé equation (1.1) and hence parametrize its solutions. We shall now describe this parametrization in detail.

Linear system (2.1) has two singular points: one irregular singular point at $\xi = \infty$ and one regular singular point at $\xi = 0$. Monodromy data associated with the point $\xi = \infty$ consist of the *Stokes matrices* S_k defined by the equation,

$$S_k = \Psi_k^{-1}(\xi)\Psi_{k+1}(\xi) \qquad k \in \mathbb{Z} \tag{2.6}$$

where $\Psi_k(\xi)$ denote the corresponding canonical solutions of system (2.1). The solutions $\Psi_k(\xi)$ are uniquely determined by the following asymptotic conditions:

$$\begin{aligned} &\Psi_k(\xi) = (I + \mathcal{O}(\xi^{-1}))\mathrm{e}^{\theta\sigma_3} \qquad \theta = \tfrac{1}{8}\xi^4 + \tfrac{1}{2}x\xi^2 + (\alpha - \beta)\ln\xi \\ &\xi \to \infty \qquad \xi \in \omega_k = \left\{\xi \in \mathbb{C} : \arg\xi \in \left(-\frac{3\pi}{8} + \frac{\pi}{4}k; \frac{\pi}{8} + \frac{\pi}{4}k\right)\right\} \qquad k \in \mathbb{Z}. \end{aligned} \tag{2.7}$$

We notice that

$$S_{2k-1} = \begin{pmatrix} 1 & s_{2k-1} \\ 0 & 1 \end{pmatrix} \qquad S_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} & 1 \end{pmatrix}$$

and the complex parameters s_k are called *Stokes multipliers*.

Besides the Stokes matrices, the monodromy data of (2.1) include the *connection matrix* E, which is defined by the equation,

$$\Psi_1(\xi) = \Psi^0(\xi)E \qquad E = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det E = 1 \tag{2.8}$$

where $\Psi^0(\xi)$ denotes the canonical solution near the regular singular point $\xi = 0$. Assuming hereafter that

$$\tfrac{1}{2} - \alpha \notin \mathbb{Z} \tag{2.9}$$

(the generic case), the solution $\Psi^0(\xi)$ is given by the equation,

$$\Psi^0(\xi) = \hat{\Psi}(\xi)\xi^{\alpha\sigma_3} \tag{2.10}$$

where $\hat{\Psi}(\xi)$ is holomorphic and invertible in the neighbourhood of $\xi = 0$, and

$$\hat{\Psi}(0) = \exp\left(\int^x uv\,\mathrm{d}x\,\sigma_3\right). \tag{2.11}$$

We note that the function $\Psi^0(\xi)$ is defined by equations (2.10) and (2.11) up to the right matrix multiplier C^{σ_3}, where C is an arbitrary non-zero complex constant. This means that the connection matrix E is defined up to the left multiplication by the constant diagonal matrix C^{σ_3}.

In the general case when $u(x), v(x), u'(x), v'(x)$ are just four arbitrary smooth functions, the monodromy data $\{S_k, E\}$ depend on x. The monodromy data do not depend on x iff

$$u' = \frac{\mathrm{d}u}{\mathrm{d}x} \qquad v' = \frac{\mathrm{d}v}{\mathrm{d}x}$$

and the functions u, v satisfy system (2.3). In other words, all the Stokes multipliers s_k and the products ac, bd (see (2.8)) are the first integrals of the nonlinear system (2.3).

The Stokes matrices and multipliers satisfy certain general constraints. In fact, the set of matrix solutions of equation (2.1) admits the symmetry automorphism,

$$\Psi(\xi) \to \sigma_3\Psi(-\xi). \tag{2.12}$$

Applying this automorphism to the canonical solutions at $\xi = \infty$, one obtains the equations

$$S_{k+4} = \mathrm{e}^{-\mathrm{i}\pi(\alpha-\beta)\sigma_3}\sigma_3 S_k \sigma_3 \mathrm{e}^{\mathrm{i}\pi(\alpha-\beta)\sigma_3} \tag{2.13}$$

or

$$s_{k+4} = -s_k \mathrm{e}^{(-1)^k 2\pi\mathrm{i}(\alpha-\beta)}. \tag{2.14}$$

Simultaneously, from (2.12) and (2.10), (2.11) it follows that

$$\sigma_3 \Psi^0(\mathrm{e}^{\mathrm{i}\pi}\xi)\sigma_3 = \Psi^0(\xi)\mathrm{e}^{\mathrm{i}\pi\alpha\sigma_3}. \tag{2.15}$$

The combination of equations (2.6), (2.15), and (2.8) implies the so-called semicyclic relation:

$$S_1 S_2 S_3 S_4 = E^{-1}\sigma_3 \mathrm{e}^{-\mathrm{i}\pi\alpha\sigma_3} E \mathrm{e}^{\mathrm{i}\pi(\alpha-\beta)\sigma_3}\sigma_3. \tag{2.16}$$

This relation leads, in particular, to the equation

$$((1 + s_1 s_2)(1 + s_3 s_4) + s_1 s_4)\mathrm{e}^{-\mathrm{i}\pi(\alpha-\beta)} - (1 + s_2 s_3)\mathrm{e}^{\mathrm{i}\pi(\alpha-\beta)} = -2\mathrm{i}\sin\pi\alpha \tag{2.17}$$

which, together with (2.14), indicates that only three of the Stokes multipliers are independent. For instance, under the generic conditions,

$$s_1 + s_{-1} + s_1 s_{-1} s_0 \neq 0 \tag{2.18}$$

the triple $\{s_{-1}, s_0, s_1\}$ form the coordinates on the manifold (2.17).

Given $\alpha, \beta \in \mathbb{C}$, equation (2.17) describes a hypersurface in the space $\mathbb{C}^4$. From (2.16) it follows that for the generic case (2.9) all essential parameters of the connection matrix E are uniquely determined by s_k (cf the next section, formula (3.15)). Therefore, in the generic case the monodromy data manifold can be identified with the surface (2.17), and any three independent Stokes multipliers, e.g. s_{-1}, s_0, s_1, form a complete set of parameters for the total set of monodromy data.

Under the gauge transformation,

$$\Psi \mapsto \mathrm{e}^{\kappa\sigma_3}\Psi\mathrm{e}^{-\kappa\sigma_3} \Leftrightarrow S_k \mapsto \mathrm{e}^{\kappa\sigma_3} S_k \mathrm{e}^{-\kappa\sigma_3} \tag{2.19}$$

the parameters u and v in (2.1) change to $\mathrm{e}^{2\kappa}u$ and $\mathrm{e}^{-2\kappa}v$, respectively, so that the Painlevé transcendent, $w = uv$, does not change. This means that any solution of PIV corresponds to an orbit of the one-parameter group of the gauge transformations (2.19) of the monodromy data manifold. The corresponding quotient manifold, which has dimension 2 in the generic case (2.9), yields the parametrization of the entire PIV transcendent set. In other words, the products $s_{2k-1}s_{2l}$ and the ratios $\frac{s_k}{s_m}$, with the integers k and m of the same parity, are the first integrals of the PIV equation. In the generic case, any (independent) two of them can be taken as universal parameters of the PIV transcendent.

Therefore, in the generic case (2.9) and under the generic conditions (2.18) the map,

$$\{\alpha, \beta, w, w'\} \mapsto \{\alpha, \beta, s_{-1}s_0, s_1 s_0\}$$

is one-to-one and the set,

$$s = \{\alpha, \beta, s_{-1}s_0, s_1 s_0\} \tag{2.20}$$

can be chosen as the monodromy parametrizations of the solutions of the PIV equation (1.1).

Assuming that x is real and $u(x), v(x), u'(x), v'(x)$ are the arbitrary real-valued functions, the space of matrix solutions of equation (2.1) admits the additional automorphism,

$$\Psi(\xi) \mapsto \sigma_3\bar{\Psi}(\bar{\xi}) \tag{2.21}$$

which in turn implies the extra symmetry equations for the Stokes matrices,

$$S_0 = \sigma_3 \bar{S}_0^{-1} \sigma_3 \qquad S_1 = \sigma_3 \bar{S}_{-1}^{-1} \sigma_3. \tag{2.22}$$

Hence, the real (for real x) solutions of (2.3) correspond to the additional restrictions on the monodromy data,

$$\bar{s}_0 = s_0 \qquad \bar{s}_{-1} = s_1. \tag{2.23}$$

The reality condition for the functions $w(x), w'(x)$ is equivalent to the weaker than (2.23) equation,

$$\overline{s_{-1}s_0} = s_1 s_0. \tag{2.24}$$

Therefore, given the real $\alpha, \beta \in \mathbb{R}$, and $\frac{1}{2} - \alpha \notin \mathbb{Z}$ the complex parameter,

$$s_- = 1 + s_1 s_0 \tag{2.25}$$

is enough to parametrize the real (for real x) solutions of the fourth Painlevé equation (1.1).

As we will see in the next sections, the generic Clarkson–McLeod one-parametric family of real solutions of (1.1) corresponds to the following specifications of the monodromy set s:

$$(1 - s_-)\mathrm{e}^{\pi \mathrm{i} \alpha} \in \mathbb{R} \qquad \beta = 0. \tag{2.26}$$

We note that under the generic condition (cf (2.18)),

$$|s_-| \neq 1 \tag{2.27}$$

restriction (2.26) and semicyclic relation (2.17) imply equations,

$$s_1 + s_3 = 0 \qquad \text{and} \qquad s_2 = 0.$$

Besides the gauge transformation, the group of *Schlesinger transformations* can be defined on the Ψ-function set. The action of this group preserves all the monodromy data except the formal monodromy exponents, i.e. the parameters α and β, and yields the *Bäcklund transformations* of the corresponding solutions of the PIV equation (1.1). We indicate specifically the following two Schlesinger transformations:

$$\tilde{\Psi} = R^{(0)} \Psi \qquad \text{and} \qquad \tilde{\Psi} = R^{(\infty)} \Psi$$

where

$$R^{(0)} = I + \frac{\mathrm{i}}{\xi} \frac{1 + 2\alpha}{2xv - v'} \sigma_+$$

and

$$R^{(\infty)} = \begin{pmatrix} \xi & \mathrm{i}u \\ -\frac{1}{\mathrm{i}u} & 0 \end{pmatrix}.$$

The corresponding Bäcklund transformations are given by the equations,

$$\tilde{w} = w + \frac{2(1 + 2\alpha)w}{w' - (w^2 + 2xw + \beta)} \qquad \tilde{\alpha} = -\alpha - 1 \qquad \tilde{\alpha} - \tilde{\beta} = \alpha - \beta \tag{2.28}$$

and

$$\tilde{w} = \frac{w'}{2w} - \frac{w}{2} - x + \frac{\beta}{2w} \qquad \tilde{\alpha} = -\alpha \qquad \tilde{\alpha} - \tilde{\beta} = \alpha - \beta + 1 \tag{2.29}$$

respectively. Transformation (2.28) is due to Kitaev [8] and transformation (2.29) is due to Lukashevich [17]. (For a comprehensive exposition of the theory of Bäcklund transformations of the PIV equation and for a detailed historical review of the subject we refer the reader to [18].)

The theory of systems of linear ODEs with rational coefficients, and particularly the complex WKB method, provides a tool for analysing the *direct monodromy problem* for system (2.1), i.e. the map,

$$w \mapsto s. \tag{2.30}$$

To put the corresponding *inverse monodromy problem*, i.e. the inverse map,

$$s \mapsto w \tag{2.31}$$

into a proper analytical context, let us introduce a piecewise analytic matrix function $\Psi(\xi)$ on the complex plane ξ, which coincides with the function $\Psi_k(\xi)$ in the closed sector

$$\Omega_k = \left\{\xi \in \mathbb{C} : \frac{\pi(k-1)}{4} \leqslant \arg\xi \leqslant \frac{\pi k}{4}\right\} \qquad k = 1, 2, \ldots, 8. \tag{2.32}$$

The function $\Psi(\xi)$ has the following characteristic properties:

(1) In the neighbourhood of $\xi = \infty$, the function $\Psi(\xi)$ satisfies the asymptotic condition given by the equation (cf equation (2.7)),

$$\Psi(\xi) = (I + \mathcal{O}(\xi^{-1}))\mathrm{e}^{\theta\sigma_3} \qquad \theta = \tfrac{1}{8}\xi^4 + \tfrac{1}{2}x\xi^2 + (\alpha - \beta)\ln\xi. \tag{2.33}$$

(2) In the neighbourhood of $\xi = 0$, the function $\Psi(\xi)$ admits the representation given by the equation (cf equations (2.8), (2.10)),

$$\Psi(\xi) = \hat{\Psi}(\xi)\xi^{\alpha\sigma_3}E(\xi) \tag{2.34}$$

where $\hat{\Psi}(\xi)$ is holomorphic and invertible in the neighbourhood of $\xi = 0$, and $E(\xi)$ is the piecewise constant matrix:

$$E(\xi) = E \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc = 1 \qquad \arg\xi \in \Omega_1$$
$$E(\xi) = ES_1 \ldots S_{k-1} \qquad \xi \in \Omega_k \qquad k = 1, 2, \ldots, 8.$$

(3) On the rays $\gamma_k = \{\xi \in \mathbb{C} : \arg\xi = \frac{\pi}{4}k\}$, $k = 1, \ldots, 8$, oriented from zero to infinity, the function $\Psi(\xi)$ has jumps given by the equations (cf equation (2.6)),

$$\begin{aligned} \Psi_+(\xi) &= \Psi_-(\xi)S_k \qquad \xi \in \gamma_k \qquad k = 1, \ldots, 7 \\ \Psi_+(\xi) &= \Psi_-(\xi)S_8\mathrm{e}^{-2\pi\mathrm{i}(\alpha-\beta)\sigma_3} \qquad \xi \in \gamma_8 \end{aligned} \tag{2.35}$$

where the symbols Ψ_+ and Ψ_- denote the limits of the function Ψ on the rays γ_k from the left and from the right, respectively.

The branches of the functions ξ^α and $\ln\xi$ are fixed by the condition,

$$0 \leqslant \arg\xi \leqslant 2\pi.$$

It is worth noticing that equation (2.11) does not need to be added in (2.34); it follows from properties (1)–(3) of the Ψ-function.

The inverse monodromy problem (2.31) for system (2.1) is equivalent (in the generic case (2.9)) to the following *Riemann–Hilbert factorization problem*: given Stokes matrices S_k and connection matrix E satisfying conditions (2.13)–(2.16), find the piecewise analytic function $\Psi(\xi)$ having properties (2.33)–(2.35). The Riemann–Hilbert problem is depicted in figure 1.

The solution $\Psi(\xi)$ of the Riemann–Hilbert problem (2.33)–(2.35) is unique (if it exists) and satisfies the symmetry equation,

$$\Psi(\xi) = \begin{cases} \sigma_3\Psi(-\xi)\sigma_3\mathrm{e}^{-\mathrm{i}\pi(\alpha-\beta)\sigma_3} & \text{for } \operatorname{Im}\xi \geqslant 0 \\ \sigma_3\Psi(-\xi)\sigma_3\mathrm{e}^{\mathrm{i}\pi(\alpha-\beta)\sigma_3} & \text{for } \operatorname{Im}\xi \leqslant 0. \end{cases} \tag{2.36}$$

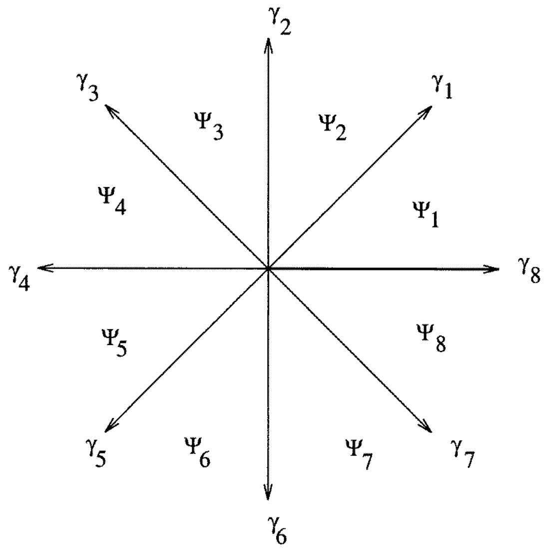

Figure 1. The Riemann–Hilbert graph γ for the Ψ-function.

Also, since the matrices S_k, E depend neither on ξ nor x, one concludes that the logarithmic derivatives, $\Psi_\xi(\xi)\Psi(\xi)^{-1}$ and $\Psi_x(\xi)\Psi(\xi)^{-1}$, are rational functions of ξ. More exactly, taking into account the asymptotic conditions (2.33), (2.34) at $\xi = \infty, 0$ and the symmetry relation (2.36) it follows that (cf [19])

$$\Psi_\xi(\xi)\Psi(\xi)^{-1} = \tfrac{1}{2}\xi^3\sigma_3 + A_2\xi^2 + A_1\xi + A_0 + A_{-1}\xi^{-1} \tag{2.37}$$

and

$$\Psi_x(\xi)\Psi(\xi)^{-1} = \tfrac{1}{2}\xi^2\sigma_3 + B_1\xi + B_0 \tag{2.38}$$

with the matrix coefficients A_k and B_k indicated in (2.1) and (2.2), respectively, and the functions $u(x), v(x)$ given via the asymptotics of $\Psi(\xi)$ as $\xi \to \infty$:

$$\Psi(\xi) = \left(I + \frac{1}{\xi}(-\mathrm{i}u\sigma_+ + \mathrm{i}v\sigma_-) + \mathcal{O}(\xi^{-2})\right)\mathrm{e}^{\theta\sigma_3}. \tag{2.39}$$

This provides the formula,

$$\begin{aligned} w(x,s) &= m_{12}m_{21} \\ m &= \lim_{\xi\to\infty}[\xi(\Psi(\xi)\mathrm{e}^{-\theta} - I)] \end{aligned} \tag{2.40}$$

for the solution of the Painlevé equation (1.1) corresponding to the given monodromy data s. Alternatively, one can use the equation (cf (2.11)),

$$w(x,s) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\hat{\Psi}_{11}(0). \tag{2.41}$$

In the next three sections, we will prove the solvability of the Riemann–Hilbert problem (2.33)–(2.35) for sufficiently large $|x|$, $x \in \mathbb{R}$ and under the assumptions (cf (2.26))

$$\begin{aligned} &\beta = 0 \qquad \alpha \in \mathbb{R} \qquad \alpha - \tfrac{1}{2} \notin \mathbb{Z} \\ &\bar{s}_0 = s_0 \qquad \bar{s}_{-1} = s_1 \\ &0 < |s_-| < 1 \qquad (1 - s_-)\mathrm{e}^{\pi\mathrm{i}\alpha} \in \mathbb{R} \end{aligned} \tag{2.42}$$

on the monodromy data. In fact we will take this further. We will obtain an *explicit* asymptotic solution of this problem, which will enable us to derive the connection formulae (1.11), (1.12) announced in theorem 1.1 and eventually prove the theorem itself.

It is worth noticing that the inequality

$$0 < |s_-| < 1$$

implies inequality (2.27) so that assumptions (2.42) yield the equations,

$$s_2 = s_1 + s_3 = 0 \qquad s_1 s_0 \neq 0.$$

It is in fact for these, weaker than (2.42) restrictions, that we will prove in the next section the solvability of the problem (2.33)–(2.35) for sufficiently large positive x. The full constraint (2.42) will be needed in sections 4 and 5 where we analyse the case of negative x.

We conclude this section by referring to [19, 12] where the solvability of the Riemann–Hilbert problems which appear in the modern theory of the Painlevé equations, and which are similar to the problem (2.33)–(2.35), are discussed in the general setting.

3. Solution of the inverse monodromy problem: $x \to +\infty$

In this section, we shall prove the following theorem.

Theorem 3.1. Suppose that $s_2 = 0$, $s_1 + s_3 = 0$, $s_1 s_0 \neq 0$, $\beta = 0$, and $\alpha - \frac{1}{2} \notin \mathbb{Z}$. Then for sufficiently large positive x, the inverse monodromy problem for system (2.1) is uniquely solvable, and the corresponding solution $w(x)$ of the PIV equation (1.1) possesses the following asymptotic behaviour as $x \to +\infty$:

$$w(x) = -\frac{s_1 s_0}{\pi^{3/2}} \mathrm{e}^{\mathrm{i}\pi\alpha} \Gamma\left(\frac{1}{2} - \alpha\right) 2^{\alpha - \frac{3}{2}} \mathrm{e}^{-x^2 + (2\alpha - 1)\ln x} (1 + \mathcal{O}(x^{-1})). \tag{3.1}$$

Proof. The proof is based on the asymptotic solution of the matrix Riemann–Hilbert problem (2.33)–(2.35) via the Deift–Zhou nonlinear steepest descent method [10]. The restrictions on the Riemann–Hilbert (monodromy) data s assumed in the theorem make the use of the method especially convenient.

Assuming $x > 0$, we can perform the scaling transformation,

$$\xi \mapsto x^{1/2}\xi \tag{3.2}$$

and

$$\Psi \mapsto x^{-\frac{\alpha-\beta}{2}\sigma_3}\Psi. \tag{3.3}$$

We shall keep the old notation, $\Psi(\xi)$, for the new Ψ-function so that the asymptotic condition (2.33) should be replaced by the condition,

$$\Psi(\xi) = (I + \mathcal{O}(\xi^{-1}))\mathrm{e}^{\theta\sigma_3} \qquad \theta = x^2\theta_0 + (\alpha - \beta)\ln\xi \qquad \theta_0 = \tfrac{1}{8}\xi^4 + \tfrac{1}{2}\xi^2 \tag{3.4}$$

and equations (2.40) and (2.41) should be replaced by the equations,

$$\begin{aligned} w(x, s) &= x m_{12} m_{21} \\ m &= \lim_{\xi \to \infty} [\xi(\Psi(\xi)\mathrm{e}^{-\theta} - I)] \end{aligned} \tag{3.5}$$

and

$$w(x, s) = -\frac{\beta}{2x} + \frac{\mathrm{d}}{\mathrm{d}x} \ln \hat{\Psi}_{11}(0) \tag{3.6}$$

respectively.

Our aim now is the asymptotic solution of the Riemann–Hilbert problem (3.4), (2.34), (2.35) under the assumptions,

$$s_2 = 0 \qquad s_1 + s_3 = 0 \qquad s_1 s_0 \neq 0 \qquad \beta = 0 \tag{3.7}$$

and

$$x \to +\infty.$$

We start with observing that the equations,

$$s_2 = 0 \qquad s_1 + s_3 = 0$$

imply that the jump matrices S_k satisfy the relations

$$S_2 = S_6 = I \qquad S_1 S_2 S_3 = S_5 S_6 S_7 = I \tag{3.8}$$

and therefore

$$\Psi_2(\xi) = \Psi_3(\xi) \qquad \Psi_6(\xi) = \Psi_7(\xi) \qquad \Psi_4(\xi) = \Psi_1(\xi) \qquad \Psi_8(\xi) = \Psi_5(\xi). \tag{3.9}$$

This means that the Riemann–Hilbert problem (2.33)–(2.35) is equivalent to the problem on the contour (cf [10]),

$$\gamma_4 \cup \gamma_8 \cup \hat{\gamma}_1 \cup \hat{\gamma}_5 \tag{3.10}$$

which is shown in figure 2. The corresponding jump conditions are:

$$\begin{aligned}
&\Psi_+ = \Psi_- S_1 \qquad S_1 = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \qquad \xi \in \hat{\gamma}_1 \\
&\Psi_+ = \Psi_- S_4 \qquad S_4 = \begin{pmatrix} 1 & 0 \\ s_4 & 1 \end{pmatrix} \qquad \xi \in \gamma_4 \\
&\Psi_+ = \Psi_- S_5 \qquad S_5 = \begin{pmatrix} 1 & -s_1 \mathrm{e}^{-2\pi \mathrm{i}\alpha} \\ 0 & 1 \end{pmatrix} \qquad \xi \in \hat{\gamma}_5 \\
&\Psi_+ = \Psi_- S_8 \mathrm{e}^{-2\pi \mathrm{i}\alpha\sigma_3} \qquad S_8 \mathrm{e}^{-2\pi \mathrm{i}\alpha\sigma_3} = \begin{pmatrix} \mathrm{e}^{-2\pi \mathrm{i}\alpha} & 0 \\ -s_4 & \mathrm{e}^{2\pi \mathrm{i}\alpha} \end{pmatrix} \qquad \xi \in \gamma_8.
\end{aligned} \tag{3.11}$$

The curves $\hat{\gamma}_1$ and $\hat{\gamma}_5$ are the *steepest-descent contours* of the exponent θ_0, i.e.

$$\operatorname{Im} \theta_0(\xi) = 0 \qquad \xi \in \hat{\gamma}_{1,5} \tag{3.12}$$

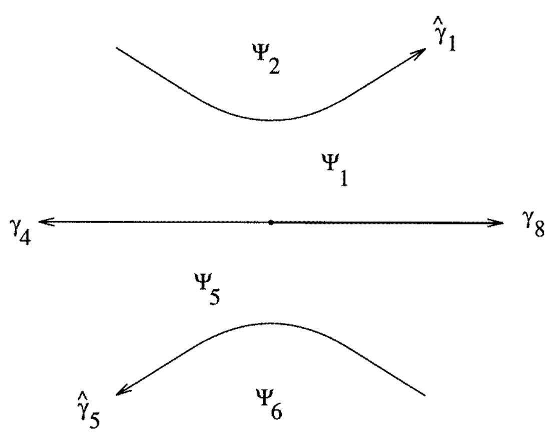

Figure 2. The Riemann–Hilbert graph for the degenerated problem $s_2 = 0$, $s_1 + s_3 = 0$.

which are asymptotic to the rays γ_1, γ_3 and γ_5, γ_7, respectively. Orientation of the curves $\hat{\gamma}_1$ and $\hat{\gamma}_5$ coincide with the orientation of the rays γ_1 and γ_5, respectively (see figure 2). In addition, the curve $\hat{\gamma}_1$ passes through the saddle point ($\frac{\mathrm{d}}{\mathrm{d}\xi}\theta_0(\xi) = 0$),

$$\xi_1 = \mathrm{i}\sqrt{2}$$

while the curve $\hat{\gamma}_5$ passes through the saddle point

$$\xi_2 = -\mathrm{i}\sqrt{2}.$$

Simultaneously, under assumptions (3.7), the semicyclic relation (2.16) takes a very simple form,

$$S_4 = E^{-1}\sigma_3 \mathrm{e}^{-\mathrm{i}\pi\alpha\sigma_3} E \mathrm{e}^{\mathrm{i}\pi\alpha\sigma_3}\sigma_3 \tag{3.13}$$

or

$$\begin{pmatrix} 1 & 0 \\ s_4 & 1 \end{pmatrix} = \begin{pmatrix} ad + bc\mathrm{e}^{2\mathrm{i}\pi\alpha} & -2bd\cos\pi\alpha \cdot \mathrm{e}^{-\mathrm{i}\pi\alpha} \\ -ac(\mathrm{e}^{2\mathrm{i}\pi\alpha} + 1) & ad + bc\mathrm{e}^{-2\mathrm{i}\pi\alpha} \end{pmatrix} \tag{3.14}$$

and hence

$$b = 0 \qquad ad = 1 \qquad s_4 = -ac(\mathrm{e}^{2\mathrm{i}\pi\alpha} + 1).$$

Therefore, we arrive to the following representation for the connection matrix E:

$$E = a^{\sigma_3} \begin{pmatrix} 1 & 0 \\ -\frac{s_4}{\mathrm{e}^{2\pi\mathrm{i}\alpha}+1} & 1 \end{pmatrix} \tag{3.15}$$

and equation (2.34) for the reduced Riemann–Hilbert problem (3.11) assumes the form,

$$\begin{aligned} \Psi(\xi) &= \hat{\Psi}(\xi)\xi^{\alpha\sigma_3} \begin{pmatrix} 1 & 0 \\ -\frac{s_4}{\mathrm{e}^{2\pi\mathrm{i}\alpha}+1} & 1 \end{pmatrix} \qquad \arg\xi \in [0;\pi] \\ \Psi(\xi) &= \hat{\Psi}(\xi)\xi^{\alpha\sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{s_4}{\mathrm{e}^{2\pi\mathrm{i}\alpha}+1}\mathrm{e}^{2\pi\mathrm{i}\alpha} & 1 \end{pmatrix} \qquad \arg\xi \in [\pi; 2\pi]. \end{aligned} \tag{3.16}$$

Let us define a new function, $\Phi(\xi)$, by the equation,

$$\Phi(\xi) = \Psi(\xi)\mathrm{e}^{-\theta\sigma_3}. \tag{3.17}$$

In terms of $\Phi(\xi)$, the Riemann–Hilbert problem (3.11), (3.16) can be rewritten as the following set of conditions.

(1)

$$\Phi(\xi) \to I \qquad \xi \to \infty \tag{3.18}$$

(2)

$$\begin{aligned} \Phi_+ &= \Phi_- G_1 \qquad G_1 = \begin{pmatrix} 1 & s_1\xi^{2\alpha}\mathrm{e}^{2x^2\theta_0} \\ 0 & 1 \end{pmatrix} \qquad \xi \in \hat{\gamma}_1 \\ \Phi_+ &= \Phi_- G_4 \qquad G_4 = \begin{pmatrix} 1 & 0 \\ s_4\xi^{-2\alpha}\mathrm{e}^{-2x^2\theta_0} & 1 \end{pmatrix} \qquad \xi \in \gamma_4 \\ \Phi_+ &= \Phi_- G_5 \qquad G_5 = \begin{pmatrix} 1 & -s_1(\mathrm{e}^{-\mathrm{i}\pi}\xi)^{2\alpha}\mathrm{e}^{2x^2\theta_0} \\ 0 & 1 \end{pmatrix} \qquad \xi \in \hat{\gamma}_5 \\ \Phi_+ &= \Phi_- G_8 \qquad G_8 = \begin{pmatrix} 1 & 0 \\ -s_4(\mathrm{e}^{-\mathrm{i}\pi}\xi_-)^{-2\alpha}\mathrm{e}^{-2x^2\theta_0} & 1 \end{pmatrix} \qquad \xi \in \gamma_8. \end{aligned} \tag{3.19}$$

(3)

$$
\begin{aligned}
&\Phi(\xi) = \hat{\Phi}(\xi)\begin{pmatrix} 1 & 0 \\ -\frac{s_4}{\mathrm{e}^{2\pi\mathrm{i}\alpha}+1}\xi^{-2\alpha}\mathrm{e}^{-2x^2\theta_0} & 1 \end{pmatrix} \\
&\arg\xi \in [0;\pi] \\
&\Phi(\xi) = \hat{\Phi}(\xi)\begin{pmatrix} 1 & 0 \\ \frac{s_4}{\mathrm{e}^{2\pi\mathrm{i}\alpha}+1}(\mathrm{e}^{-\pi\mathrm{i}}\xi)^{-2\alpha}\mathrm{e}^{-2x^2\theta_0} & 1 \end{pmatrix} \\
&\arg\xi \in [\pi;2\pi].
\end{aligned}
\tag{3.20}
$$

It should be emphasized that no asymptotic analysis has been made so far. The Riemann–Hilbert problem (3.18)–(3.20) is just a reformulation of the original problem (2.33)–(2.35) under the assumptions, $s_2 = s_1 + s_3 = 0$, $\beta = 0$. The main advantage of this reformulation, besides the I-normalization of the asymptotic condition at $\xi = \infty$, is that the Φ-problem is posed on the steepest-descent curves of the exponent θ_0 so that all the jump matrices approach exponentially the identity as $x \to \infty$.

Our next (and the final) step is the asymptotic solution of the problem (3.18)–(3.20). The basic idea is to approximate the exact solution $\Phi(\xi)$ by the product,

$$\Phi_0(\xi) \equiv Y(\xi)X(\xi)$$

where the matrix functions $X(\xi)$ and $Y(\xi)$ are the solutions of the model Riemann–Hilbert problems related to the contours $\hat{\gamma}_1 \cup \hat{\gamma}_5$ and $\gamma_4 \cup \gamma_8 = \mathbb{R}$, respectively. More exactly, the functions $Y(\xi)$ and $X(\xi)$ are determined by the conditions:

(1) $Y(\xi)$ is analytic in $\mathbb{C}\backslash\hat{\gamma}_1 \cup \hat{\gamma}_5$,

(2) $Y(\xi) \to I$ as $\xi \to \infty$,

(3)

$$
\begin{aligned}
&Y_+ = Y_- G_1 \qquad \xi \in \hat{\gamma}_1 \\
&Y_+ = Y_- G_5 \qquad \xi \in \hat{\gamma}_5
\end{aligned}
\tag{3.21}
$$

and

(1) $X(\xi)$ is analytic in $\mathbb{C}\backslash\gamma_4 \cup \gamma_8$,

(2) $X(\xi) \to I$ as $\xi \to \infty$,

(3)

$$
\begin{aligned}
&X_+ = X_- G_4 \qquad \xi \in \gamma_4 \\
&X_+ = X_- G_8 \qquad \xi \in \gamma_8
\end{aligned}
\tag{3.22}
$$

(4)

$$
\begin{aligned}
&X(\xi) = \hat{X}(\xi)\begin{pmatrix} 1 & 0 \\ -\frac{s_4}{\mathrm{e}^{2\pi\mathrm{i}\alpha}+1}\xi^{-2\alpha}\mathrm{e}^{-2x^2\theta_0} & 1 \end{pmatrix} \qquad \arg\xi \in [0;\pi] \\
&X(\xi) = \hat{X}(\xi)\begin{pmatrix} 1 & 0 \\ \frac{s_4}{\mathrm{e}^{2\pi\mathrm{i}\alpha}+1}(\mathrm{e}^{-\pi\mathrm{i}}\xi)^{-2\alpha}\mathrm{e}^{-2x^2\theta_0} & 1 \end{pmatrix} \qquad \arg\xi \in [\pi;2\pi].
\end{aligned}
\tag{3.23}
$$

Assume temporarily that

$$\operatorname{Re}\alpha < \tfrac{1}{2}. \tag{3.24}$$

Then both the model problems, (3.21) and (3.22), can be solved explicitly in terms of the Cauchy integrals:

$$Y(\xi) = I + y(\xi)\sigma_+ \qquad X(\xi) = I + h(\xi)\sigma_- \tag{3.25}$$

where

$$y(\xi) = \frac{s_1}{2\pi\mathrm{i}}\int_{\hat{\gamma}_1}\frac{\tau^{2\alpha}\mathrm{e}^{2x^2\theta_0(\tau)}}{\tau-\xi}\,\mathrm{d}\tau - \frac{s_1}{2\pi\mathrm{i}}\int_{\hat{\gamma}_5}\frac{(\mathrm{e}^{-\mathrm{i}\pi}\tau)^{2\alpha}\mathrm{e}^{2x^2\theta_0(\tau)}}{\tau-\xi}\,\mathrm{d}\tau$$
$$= \frac{s_1}{\pi\mathrm{i}}\xi\int_{\hat{\gamma}_1}\frac{\tau^{2\alpha}\mathrm{e}^{2x^2\theta_0(\tau)}}{\tau^2-\xi^2}\,\mathrm{d}\tau \tag{3.26}$$

$$h(\xi) = \frac{s_4}{2\pi\mathrm{i}}\int_{\gamma_4}\frac{\tau^{-2\alpha}\mathrm{e}^{-2x^2\theta_0(\tau)}}{\tau-\xi}\,\mathrm{d}\tau - \frac{s_4}{2\pi\mathrm{i}}\int_{\gamma_8}\frac{(\mathrm{e}^{-\mathrm{i}\pi}\tau_-)^{-2\alpha}\mathrm{e}^{-2x^2\theta_0(\tau)}}{\tau-\xi}\,\mathrm{d}\tau$$
$$= \frac{s_4}{\pi\mathrm{i}}\xi\int_{\gamma_4}\frac{\tau^{-2\alpha}\mathrm{e}^{-2x^2\theta_0(\tau)}}{\tau^2-\xi^2}\,\mathrm{d}\tau \tag{3.27}$$

and the integrals in (3.27) are well defined due to assumption (3.24).

Now let $R(\xi)$ be a matrix ratio,

$$R(\xi) = \Phi(\xi)[\Phi_0(\xi)]^{-1} = \Phi(\xi)X(\xi)^{-1}Y(\xi)^{-1}. \tag{3.28}$$

A comparison of equations (3.19), (3.20) and (3.22), (3.23) shows that $R(\xi)$ has no jumps and singularities on the real axis, including $\xi = 0$, but still have jumps on the contour $\hat{\gamma}_1 \cup \hat{\gamma}_5$ where it solves the following Riemann–Hilbert problem:

(1) $R(\xi) \to I$ as $\xi \to \infty$,

(2)

$$R_+ = R_- G_0 \tag{3.29}$$

where

$$G_0 = Y_- X G_{1,5} X^{-1} G_{1,5}{}^{-1} Y_-{}^{-1} \qquad \xi \in \hat{\gamma}_{1,5}. \tag{3.30}$$

The curves γ_4, $\hat{\gamma}_1$, and $\hat{\gamma}_5$ are the steepest-descent contours for the exponent $\theta_0(\xi)$ (see (3.12)) so that the integral representations (3.26) and (3.27) lead to the uniform estimates,

$$|y_-(\xi)| < \frac{C}{|\xi|}\mathrm{e}^{-x^2} \tag{3.31}$$

and

$$|h(\xi)| < \frac{C}{|\xi|}x^{2\mathrm{Re}\,\alpha-1} \tag{3.32}$$

for all $x > 1$ and $\xi \in \hat{\gamma}_1 \cup \hat{\gamma}_5$. Rewriting equation (3.30) componentwise, we derive from (3.31), (3.32) the inequalities,

$$\|I - G_0^{-1}(\xi)\| < C|\xi|^{2\mathrm{Re}\,\alpha-1}x^{2\mathrm{Re}\,\alpha-1}\mathrm{e}^{2x^2\theta_0(\xi)} \tag{3.33}$$

and

$$|(G_0^{-1}(\xi))_{12}| < C|\xi|^{2\mathrm{Re}\,\alpha-1}x^{2\mathrm{Re}\,\alpha-1}\mathrm{e}^{-x^2+2x^2\theta_0(\xi)} \tag{3.34}$$

which hold uniformly for all $x > 1$ and $\xi \in \hat{\gamma}_1 \cup \hat{\gamma}_5$. In addition, from (3.33) we have an estimate for the corresponding L_2-norm as well:

$$\|I - G_0^{-1}\|_{L_2(\hat{\gamma}_1\cup\hat{\gamma}_5)} < Cx^{2\mathrm{Re}\,\alpha-\frac{3}{2}}\mathrm{e}^{-x^2}. \tag{3.35}$$

We note that the actual value of the positive constant C is not important to us and may be different in different formulae.

By a standard technique in the theory of the Riemann–Hilbert problem (see e.g. [20]; see also [21, 10]), the solution $R(\xi)$ of the Riemann–Hilbert problem (3.29) is given by the formula

$$R(\xi) = I + \frac{1}{2\pi \mathrm{i}} \int_{\hat\gamma_1 \cup \hat\gamma_5} \rho(\tau) \left[I - G_0^{-1}(\tau)\right] \frac{\mathrm{d}\tau}{\tau - \xi} \tag{3.36}$$

where $\rho(\xi) \equiv R_+(\xi)$ solves the equation

$$\rho = I + C_+[\rho(I - G_0^{-1})] \tag{3.37}$$

in $L_2(\hat\gamma_1 \cup \hat\gamma_5)$), and C_+ is the corresponding Cauchy operator.

The L_2-boundness of the operator C_+ (see e.g. [22, 20]; see also [21, 23]), together with estimates (3.33), (3.35) imply the solvability of the singular integral equation (3.37) for sufficiently large positive x and the asymptotic equation,

$$\|I - \rho\|_{L_2(\hat\gamma_1 \cup \hat\gamma_5)} = \mathcal{O}(x^{2\alpha - \frac{3}{2}} \mathrm{e}^{-x^2}) \qquad x \to +\infty \tag{3.38}$$

for its solution $\rho(\xi)$.

The solvability of the singular integral equation (3.37) yields the solvability of the Riemann–Hilbert problem (3.29). Rewriting representation (3.36) for the solution $R(\xi)$ in the form,

$$R(\xi) = I + \frac{1}{2\pi \mathrm{i}} \int_{\hat\gamma_1 \cup \hat\gamma_5} [I - G_0^{-1}(\tau)] \frac{\mathrm{d}\tau}{\tau - \xi} + \frac{1}{2\pi \mathrm{i}} \int_{\hat\gamma_1 \cup \hat\gamma_5} [\rho(\tau) - I][I - G_0^{-1}(\tau)] \frac{\mathrm{d}\tau}{\tau - \xi}$$

and applying again estimates (3.33), (3.35) together with estimates (3.38) and (3.34), we conclude that the inequalities,

$$\|I - R(\xi)\| < \frac{C}{|\xi|} x^{2\mathrm{Re}\,\alpha - 2} \mathrm{e}^{-x^2} \tag{3.39}$$

and

$$|R_{12}(\xi)| < \frac{C}{|\xi|} x^{2\mathrm{Re}\,\alpha - 2} \mathrm{e}^{-2x^2} \tag{3.40}$$

take place uniformly for all $\xi \in \mathrm{i}\mathbb{R}$, $|\xi| \geqslant 2\sqrt{2}$.

The solvability for sufficiently large x of the Riemann–Hilbert problem for function $R(\xi)$ implies in turn the solvability of the basic Riemann–Hilbert problem (3.18)–(3.20) and hence the solvability, for sufficiently large positive x, of the inverse monodromy problem for system (2.1) under assumptions (3.7). Moreover, estimates (3.39), (3.40) and equations (3.5), (3.28), (3.25) lead to the following asymptotic representation for the corresponding Painlevé function $w(x)$:

$$w(x) = x(m_{12}^0 + \mathcal{O}(x^{2\alpha - 2} \mathrm{e}^{-2x^2}))(m_{21}^0 + \mathcal{O}(x^{2\alpha - 2} \mathrm{e}^{-x^2})) \tag{3.41}$$

where

$$m_{12}^0 = \frac{\mathrm{i}}{\pi} s_1 \int_{\hat\gamma_1} \tau^{2\alpha} \mathrm{e}^{2x^2 \theta_0(\tau)}\, \mathrm{d}\tau$$

and

$$m_{21}^0 = \frac{\mathrm{i}}{\pi} s_4 \int_{\gamma_4} \tau^{-2\alpha} \mathrm{e}^{-2x^2 \theta_0(\tau)}\, \mathrm{d}\tau.$$

Evaluating the last two contour integrals by the *classical* steepest-descent method, we obtain that

$$m_{12}^0 = \frac{\mathrm{i}}{\sqrt{2\pi}} s_1 2^\alpha \mathrm{e}^{\mathrm{i}\pi\alpha} \frac{\mathrm{e}^{-x^2}}{x} (1 + \mathcal{O}(x^{-1}))$$

and

$$m_{21}^{0} = \frac{1}{2\pi \mathrm{i}} s_4 \mathrm{e}^{-2\mathrm{i}\pi\alpha} x^{2\alpha-1} \Gamma\left(\frac{1}{2}-\alpha\right)(1+\mathcal{O}(x^{-1}))$$

which, in virtue of (3.41), yields the asymptotics of the Painlevé function w presented in the theorem:

$$w = \frac{s_1 s_4}{\pi^{3/2}} \mathrm{e}^{-\mathrm{i}\pi\alpha} \Gamma\left(\frac{1}{2}-\alpha\right) 2^{\alpha-\frac{3}{2}} \mathrm{e}^{-x^2+(2\alpha-1)\ln x}(1+\mathcal{O}(x^{-1})) \qquad x \to +\infty \tag{3.42}$$

(note that due to (2.14) $s_4 = -s_0 \mathrm{e}^{2\mathrm{i}\pi\alpha}$).

Asymptotics (3.42) has the form (1.3) with the parameter

$$k^2 = -s_0 s_1 \mathrm{e}^{\mathrm{i}\pi\alpha} \frac{1}{2(2\pi)^{3/2}} \Gamma\left(\frac{1}{2}-\alpha\right). \tag{3.43}$$

Moreover, if we want the asymptotics (3.42) to be consistent with the reality condition, the extra restriction,

$$s_1 s_4 \mathrm{e}^{-\mathrm{i}\pi\alpha} \equiv -s_0 s_1 \mathrm{e}^{\mathrm{i}\pi\alpha} \in \mathbb{R} \tag{3.44}$$

should be imposed on the monodromy data. This in turn yields specification (2.26) of the Clarkson–McLeod Painlevé transcendent.

In [3] it was shown that a suitable chain of the Bäcklund transformations (2.28) and (2.29), properly combined with the transformation generated by the rotation, $x \mapsto \mathrm{i}x$, preserves the Stokes multipliers s_k and the value of the parameter $\beta = 0$ and transforms $\alpha \mapsto \alpha + 1$. Moreover, the same chain of the Bäcklund transformations preserves the exponential behaviour (1.3) with the substitutions:

$$\alpha \mapsto \alpha + 1$$
$$k^2 \mapsto \frac{k^2}{\alpha + \frac{1}{2}}.$$

In view of equation (3.43), this allows us to drop the condition $\mathrm{Re}\,\alpha < \frac{1}{2}$ in formula (3.42) and hence complete the proof of theorem 3.1. □

4. Solution of the direct monodromy problem: $x \to -\infty$

Let us make the gauge transformation (2.19) with $\kappa = \frac{1}{2}\ln v - \frac{1}{4}\ln(-x)$:

$$\Psi \mapsto (-x)^{-\frac{\sigma_3}{4}} v^{\frac{\sigma_3}{2}} \Psi v^{-\frac{\sigma_3}{2}} (-x)^{\frac{\sigma_3}{4}} \tag{4.1}$$

so that the basic system (2.1) transforms to the matrix equation,

$$\begin{aligned}\frac{\mathrm{d}\Psi}{\mathrm{d}\xi} = \bigg\{ &\left(\frac{1}{2}\xi^3 + \xi(x+w) + \frac{\alpha}{\xi}\right)\sigma_3 + \mathrm{i}(-x)^{-1/2} w \left(\xi^2 + x + \frac{w'}{2w} + \frac{1}{2}w + \frac{\beta}{2w}\right)\sigma_+ \\ &+ \mathrm{i}(-x)^{1/2}\left(\xi^2 + x - \frac{w'}{2w} + \frac{1}{2}w + \frac{\beta}{2w}\right)\sigma_- \bigg\}\Psi.\end{aligned} \tag{4.2}$$

This equation belongs to the general class of systems (2.1) (u', v' are not necessarily the x-derivatives of u, v), and it is specified by the condition,

$$v = (-x)^{1/2}. \tag{4.3}$$

Condition (4.3) is not gauge invariant, and the monodromy data for system (4.2) are uniquely determined, in the generic case, by two complex parameters,

$$s_{-1}s_0 \qquad s_1s_0$$

for complex pairs $\{w, w'\}$, and by one complex parameter,

$$s_- = 1 + s_1s_0 \tag{4.4}$$

for real pairs $\{w, w'\}$. This fact, which implies the injectivity of the map,

$$\{w, w'\} \mapsto s_- \qquad (w, w' \text{ are real}) \tag{4.5}$$

has already been mentioned in section 2 (see (2.20)–(2.25)) where the monodromy theory for the systems of class (2.1) has been outlined (without the detailed proofs) according to [8]. The injectivity of map (4.5) will be especially important to us in section 5 and will be proven there for the reader's convenience.

It is also worth mentioning that for the arbitrary pair of the real-valued (for real x) functions $w(x)$, $w'(x)$, the monodromy parameter s_- depends on x. It does not depend on x iff

$$w' = \frac{\mathrm{d}w}{\mathrm{d}x}$$

and the function $w(x)$ satisfies the PIV equation (1.1). It should be emphasized that even in this case the Stokes matrices of system (4.2) *may depend on* x via the similarity transformation,

$$S_k(x) = \mathrm{e}^{\kappa(x)\sigma_3}S_k(0)\mathrm{e}^{-\kappa(x)\sigma_3}.$$

The main objective of this section is the following result.

Theorem 4.1. Let $\alpha, \beta \in \mathbb{R}$, and s_-^*, s_- be the complex numbers satisfying the conditions,

$$0 < |s_-^*| < 1 \tag{4.6}$$

and

$$s_- \in D(s_-^*; \varepsilon) = \{s_- \in \mathbb{C} : |s_- - s_-^*| \leqslant \varepsilon\} \qquad 0 < \varepsilon < \min\{1 - |s_-^*|, |s_-^*|\}. \tag{4.7}$$

Define the functions $w(x)$ and $w'(x)$ in (4.2) by the equations,

$$w \equiv \hat{w}(x, s_-) = -\frac{2x}{3} + 2\sqrt{2}a\cos\Theta \qquad w' \equiv \frac{\mathrm{d}\hat{w}(x, s_-)}{\mathrm{d}x} \tag{4.8}$$

where

$$a^2 = -\frac{1}{2\sqrt{3}\pi}\ln(1 - |s_-|^2) \qquad a > 0 \tag{4.9}$$

$$\Theta = \frac{x^2}{\sqrt{3}} - \sqrt{3}a^2\ln(2\sqrt{3}x^2) + \phi \tag{4.10}$$

$$\phi = -\frac{3\pi}{4} - \frac{2\pi}{3}(\alpha - \beta) - \arg\Gamma(-\mathrm{i}\sqrt{3}a^2) - \arg s_-$$

and denote

$$\hat{s}_-(x, s_-) \equiv 1 + \hat{s}_0\hat{s}_1(x, s_-)$$

the corresponding monodromy parameter (4.4). Then there exist real constants $C(s_-^*; \varepsilon) > 0$ and $x_0(s_-^*; \varepsilon) < -1$ such that

$$|\hat{s}_-(x, s_-) - s_-| \leqslant (-x)^{-\frac{1}{4}}C(s_-^*; \varepsilon) \tag{4.11}$$

for all

$$x < x_0(s_-^*;\varepsilon) \qquad s_- \in D(s_-^*;\varepsilon).$$

As it is indicated, the constants $C(s_-^*;\varepsilon)$ and $x_0(s_-^*;\varepsilon)$ only depend on s_-^* and ε.

Throughout this section, the dependence of all the estmates on α and β is not important to us. The crucial point is that the r.h.s. of inequality (4.11) *does not depend on* s_-, i.e. estimate (4.11) is uniform on $D(s_-^*;\varepsilon)$. This will allow us to use Kitaev's method [11] and transform in section 5 (see theorem 5.1) the above result into the rigorous statement about the asymptotic behaviour of the Painlevé transcendent $w(x,s_-)$, $0<|s_-|<1$ as $x\to-\infty$.

Proof of theorem 4.1. The change of variables,

$$\begin{aligned}&\xi = (-x)^{1/2}\lambda \qquad w=(-x)r \qquad w' = -(-x)^2\left(r'+\frac{r}{2t}\right)\\ &r'=\frac{\mathrm{d}r}{\mathrm{d}t} \qquad t=\frac{1}{2}(-x)^2 \qquad \Phi(\lambda)=\Psi(\xi(\lambda))\end{aligned} \tag{4.12}$$

brings equation (4.2) to the form:

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = 2t\left\{\left(\frac{1}{2}\lambda^3+\lambda(r-1)+\frac{\alpha}{2t\lambda}\right)\sigma_3 + \mathrm{i}r\left(\lambda^2-1+\frac{r}{2}+\frac{\beta}{4tr}-\left(\frac{r'}{2r}+\frac{1}{4t}\right)\right)\sigma_+\right.\\ \left.+\,\mathrm{i}\left(\lambda^2-1+\frac{r}{2}+\frac{\beta}{4tr}+\left(\frac{r'}{2r}+\frac{1}{4t}\right)\right)\sigma_-\right\}\Phi \equiv 2tA\Phi.\end{aligned} \tag{4.13}$$

The matrix A is already of order $O(1)$. Hence, in carring out the relevant asymptotic analysis, one can appeal to the classical WKB-method (see [24–26]).

One of the principal elements of the complex WKB-method is the eigenvalues of the system (4.13), i.e.

$$\mu_{1,2} = \pm\mu = \pm\sqrt{-\det A}$$

which are given by the equation,

$$\mu^2 = \frac{1}{4}\left(\lambda^2-\frac{8}{3}\right)\left(\lambda^2-\frac{2}{3}\right)^2 + \frac{\alpha-\beta}{2t}\lambda^2 + p + \frac{\alpha^2}{4t^2\lambda^2} \tag{4.14}$$

where

$$p = \frac{1}{4r}\left(r'+\frac{r}{2t}\right)^2 - \frac{1}{4}\left(r-\frac{8}{3}\right)\left(r-\frac{2}{3}\right)^2 - \frac{\beta-4\alpha}{4t}r - \frac{2\alpha-\beta}{2t} - \frac{\beta^2}{16t^2r}.$$

Because of assumption (4.8),

$$\begin{aligned}&r = \frac{2}{3}+\frac{2a}{\sqrt{t}}\cos\Theta\\ &r'+\frac{r}{2t} = -\frac{4a}{\sqrt{3t}}\left(1-\frac{3a^2}{2t}\right)\sin\Theta + \frac{1}{3t}\\ &\Theta = \frac{2t}{\sqrt{3}} - \sqrt{3}a^2\ln(4\sqrt{3}t)+\phi\end{aligned} \tag{4.15}$$

and we have the estimates,

$$\begin{aligned}&\tfrac{1}{3} < r < 1\\ &|p| \leqslant \frac{1}{t}c_0(a) \qquad \left|p-\frac{2a^2}{t}-\frac{\beta-\alpha}{3t}\right| \leqslant \frac{1}{t^{3/2}}c_0(a)\end{aligned} \tag{4.16}$$

for all

$$t > t_0(a) > 1 \qquad a : 0 < a_1 \leqslant a \leqslant a_2 < +\infty$$

where the positive constants $t_0(a)$ and $c_0(a)$ depend continuously on the quantity a.

System (4.13) has eight (if $\alpha \neq 0$) and six (if $\alpha = 0$) real turning points, i.e. the zeros of $\mu(\lambda)$. It follows from (4.14) and (4.16) that these points can be numerated in such way that

$$\begin{aligned}
&\left|\lambda_{1,3} - \sqrt{\frac{2}{3}}\right| \leqslant \frac{1}{\sqrt{t}} c_1(a) \qquad \lambda_3 < \sqrt{\frac{2}{3}} < \lambda_1 \qquad \lambda_{2,4} = -\lambda_{1,3} \\
&\left|\lambda_5 - \sqrt{\frac{8}{3}}\right| \leqslant \frac{1}{t} c_1(a) \qquad \lambda_6 = -\lambda_5 \\
&|\lambda_{7,8}| \leqslant \frac{1}{t} c_0(a) \qquad \lambda_8 = -\lambda_7 \qquad \lambda_7 > 0 \qquad \alpha \neq 0
\end{aligned} \tag{4.17}$$

for all

$$t > t_1(a) \qquad a : 0 < a_1 \leqslant a \leqslant a_2 < +\infty$$

and some positive $t_1(a), c_1(a)$ depending continuously on a.

The points λ_1 and λ_3 tend to $\sqrt{\frac{2}{3}}$ as $t \to \infty$, and therefore the point $\sqrt{\frac{2}{3}}$ should be considered as an asymptotically double turning point. The points $\lambda_{2,4}$ behave similarly, so that the point $-\sqrt{\frac{2}{3}}$ is another double turning point. The points λ_5 and $\lambda_6 = -\lambda_5$ are the single turning points. The turning points λ_7 and $\lambda_8 = -\lambda_7$ merge with the singularity at the point zero.

Using the continuity of the mapping,

$$s_- \mapsto a = \sqrt{-\frac{1}{2\sqrt{3}\pi} \ln(1 - |s_-|^2)}$$

and the compactness of the domain $D(s_-^*; \varepsilon)$, we conclude from (4.16) and (4.17) that there exist the positive constants $C = C(s_-^*; \varepsilon)$ and $t_0(s_-^*; \varepsilon) > 1$ such that

$$\begin{aligned}
&|p| \leqslant \frac{1}{t} C(s_-^*; \varepsilon) \\
&\left|p - \frac{2a^2}{t} - \frac{\beta - \alpha}{3t}\right| \leqslant \frac{1}{t^{3/2}} C(s_-^*; \varepsilon) \\
&\tfrac{1}{3} < r < 1 \\
&\left|\lambda_{1,3} - \sqrt{\frac{2}{3}}\right| \leqslant \frac{1}{\sqrt{t}} C(s_-^*; \varepsilon) \\
&\left|\lambda_5 - \sqrt{\frac{8}{3}}\right| \leqslant \frac{1}{t} C(s_-^*; \varepsilon) \\
&|\lambda_7| \leqslant \frac{1}{t} C(s_-^*; \varepsilon) \\
&\forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_0(s_-^*; \varepsilon).
\end{aligned} \tag{4.18}$$

Remark 4.1. Hereafter we shall assume the following convention.

The symbols C, C_j, and t_0 denote positive constants, which only depend on s_-^* and ε:

$$C = C(s_-^*; \varepsilon) \qquad C_j = C_j(s_-^*; \varepsilon) \qquad t_0 = t_0(s_-^*; \varepsilon).$$

Symbol t_1, which will appear later, denotes positive constant, which only depends on s_-^*, ε, and δ:

$$t_1 = t_1(s_-^*; \varepsilon; \delta).$$

The actual values of C, C_j, t_0, and t_1 are not important to us and may be different in different formulae.

Let us now outline the basic steps of the IM technique (cf [7, 9]) which we are going to use in our proof.

(1) Calculation of the WKB-solutions Φ_+^{WKB} and Φ_-^{WKB} associated with the double turning point $\sqrt{\frac{2}{3}}$ and related to the Stokes rays $\arg\lambda = \frac{3\pi}{8}$ and $\arg\lambda = -\frac{3\pi}{8}$, respectively.

(2) Calculation of the solution Φ^{TP} near the double turning point $\sqrt{\frac{2}{3}}$.

(3) Matching of the canonical solutions Ψ_2 and Ψ_{-1} with the WKB-solutions Φ_+^{WKB} and Φ_-^{WKB} respectively. Asymptotic evaluation of the matrices,

$$C_\pm = [\Phi_\pm^{\mathrm{WKB}}(\lambda)]^{-1}\Psi_{2,-1}(\xi(\lambda)). \tag{4.19}$$

(4) Matching of the WKB-solutions $\Phi_\pm^{\mathrm{WKB}}$ with the turning point solution Φ^{TP}. Asymptotic evaluation of the matrices,

$$N_\pm = [\Phi^{\mathrm{TP}}(\lambda)]^{-1}\Phi_\pm^{\mathrm{WKB}}(\lambda). \tag{4.20}$$

(5) Using the equation,

$$C_-^{-1}N_-^{-1}N_+C_+ = \hat{S}_{-1}\hat{S}_0\hat{S}_1 \equiv \begin{pmatrix} 1+\hat{s}_{-1}\hat{s}_0 & \hat{s}_{-1}+\hat{s}_1+\hat{s}_{-1}\hat{s}_0\hat{s}_1 \\ \hat{s}_0 & 1+\hat{s}_0\hat{s}_1 \end{pmatrix} \tag{4.21}$$

for the derivation of the asymptotic formula for the indicated product of the Stokes matrices corresponding to the coefficient function $w = \hat{w}(x, s_-)$ given in (4.8). Asymptotic evaluation of the monodromy parameter,

$$\hat{s}_- = (\hat{S}_{-1}\hat{S}_0\hat{S}_1)_{22}. \tag{4.22}$$

Technically, we are going to perform the standard WKB-type calculations (cf [7]) but by undertaking special efforts (cf [9]) to secure that all the estimates are uniform with respect to $s \in D(s_-^*; \varepsilon)$.

Step 1. WKB-approximation. Let us introduce the notations a_3, a_+, a_- for the entries of the matrix A from (4.13):

$$A = a_3\sigma_3 + a_+\sigma_+ + a_-\sigma_-. \tag{4.23}$$

Consider the gauge transformation of the matrix Φ,

$$\Phi(\lambda) = T(\lambda)Y(\lambda) \tag{4.24}$$

with the matrix T given by

$$T = \begin{pmatrix} 1 & 1 \\ h_1 & h_2 \end{pmatrix} \tag{4.25}$$

where

$$h_1 = \frac{\mu - a_3}{a_+} - \frac{1}{2t}\cdot\frac{1}{2\mu}\left(\frac{\mu - a_3}{a_+}\right)'$$
$$h_2 = -\frac{\mu + a_3}{a_+} - \frac{1}{2t}\cdot\frac{1}{2\mu}\left(\frac{\mu + a_3}{a_+}\right)'$$

and $(\ldots)' \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda}(\ldots)$. Here, $\mu^2 = a_3^2 + a_+a_-$, and its exact expression is given in (4.14). We also assume that the function $\mu(\lambda)$ is defined on the complex plane cut along the small segments $(\lambda_1;\lambda_3)$, $(\lambda_2;\lambda_4)$, $(\lambda_7;\lambda_8)$ and along the rays $(\lambda_5;+\infty)$, $(\lambda_6;-\infty)$. The branch of the root is chosen in such a way that $\mu(\lambda)\to\frac{1}{2}\lambda^3-\lambda$ as $\lambda\to+\mathrm{i}\infty$.

The new function $Y(\lambda)$ satisfies the equation

$$\frac{\mathrm{d}Y}{\mathrm{d}\lambda} = 2tBY \tag{4.26}$$

with the matrix

$$B = \begin{pmatrix} a_3 + a_+h_1 + H_1 & H_2 \\ -H_1 & a_3 + a_+h_2 - H_2 \end{pmatrix} \tag{4.27}$$

where

$$a_3 + a_+h_1 = \mu - \frac{1}{2t}\cdot\frac{a_+}{2\mu}\left(\frac{\mu-a_3}{a_+}\right)'$$

$$a_3 + a_+h_2 = -\mu - \frac{1}{2t}\cdot\frac{a_+}{2\mu}\left(\frac{\mu+a_3}{a_+}\right)'$$

$$H_1 = \frac{1}{4t^2}\frac{1}{1+\frac{a_+}{4t\mu^2}\left(\frac{a_3}{a_+}\right)'}\cdot\frac{a_+}{2\mu}\left[-a_+\left(\frac{1}{2\mu}\left(\frac{\mu-a_3}{a_+}\right)'\right)^2 + \left(\frac{1}{2\mu}\left(\frac{\mu-a_3}{a_+}\right)'\right)'\right]$$

$$H_2 = \frac{1}{4t^2}\frac{1}{1+\frac{a_+}{4t\mu^2}\left(\frac{a_3}{a_+}\right)'}\cdot\frac{a_+}{2\mu}\left[-a_+\left(\frac{1}{2\mu}\left(\frac{\mu+a_3}{a_+}\right)'\right)^2 + \left(\frac{1}{2\mu}\left(\frac{\mu+a_3}{a_+}\right)'\right)'\right].$$

Let

$$\begin{aligned}\Lambda &= \begin{pmatrix} \mu - \frac{1}{2t}\frac{a_+}{2\mu}\left(\frac{\mu-a_3}{a_+}\right)' & \\ & -\mu - \frac{1}{2t}\frac{a_+}{2\mu}\left(\frac{\mu+a_3}{a_+}\right)' \end{pmatrix} \\ &= \mu\sigma_3 + \frac{1}{4t\mu}\left(a_3' - a_3\frac{a_+'}{a_+}\right)\sigma_3 - \frac{1}{4t}\left(\frac{\mu'}{\mu} - \frac{a_+'}{a_+}\right)I \end{aligned} \tag{4.28}$$

and

$$R = B - \Lambda = \begin{pmatrix} H_1 & H_2 \\ -H_1 & -H_2 \end{pmatrix}. \tag{4.29}$$

The WKB solution of (4.26) associated with the double turning point $\sqrt{\frac{2}{3}}$ can be defined as

$$Y^{\mathrm{WKB}}(\lambda) = \chi(\lambda)\exp\left\{2t\int_{\lambda_0}^{\lambda}\Lambda\,\mathrm{d}\lambda\right\} \tag{4.30}$$

where the lower limit λ_0 can be chosen arbitrary, and the matrix function $\chi(\lambda)$ satisfies the integral equation,

$$\begin{aligned}\chi(\lambda) &= I + 2t\int_{\gamma(\lambda)} \mathrm{e}^{(2t\int_\zeta^\lambda \Lambda(z)\,\mathrm{d}z)}R(\zeta)\chi(\zeta)\mathrm{e}^{(-2t\int_\zeta^\lambda \Lambda(z)\,\mathrm{d}z)}\,\mathrm{d}\zeta \\ &= I + 2t\int_{\gamma(\lambda)} \mathrm{e}^{2t\int_\zeta^\lambda \mu(z)\,\mathrm{d}z\sigma_3}\{\mathrm{e}^{\int_\zeta^\lambda \eta(z)\,\mathrm{d}z\sigma_3}R(\zeta)\chi(\zeta)\mathrm{e}^{-\int_\zeta^\lambda \eta(z)\,\mathrm{d}z\sigma_3}\} \\ &\quad\times\mathrm{e}^{-2t\int_\zeta^\lambda \mu(z)\,\mathrm{d}z\sigma_3}\,\mathrm{d}\zeta \end{aligned} \tag{4.31}$$

$$\eta(z) = \frac{1}{2\mu}\left(a_3' - a_3\frac{a_+'}{a_+}\right).$$

Here, $\gamma(\lambda) = (\gamma_1(\lambda); \gamma_2(\lambda))$ is a matrix of the canonical paths (cf [26]), i.e. the simple contours which start at λ and end up at ∞, and which satisfy the conditions:

$$\begin{aligned} &\operatorname{Re}\int_{\zeta}^{\lambda} \mu(z)\,\mathrm{d}z \uparrow +\infty \qquad \zeta \to \infty \qquad \zeta \in \gamma_1(\lambda) \\ &\operatorname{Re}\int_{\zeta}^{\lambda} \mu(z)\,\mathrm{d}z \downarrow -\infty \qquad \zeta \to \infty \qquad \zeta \in \gamma_2(\lambda). \end{aligned} \tag{4.32}$$

Matrix equation (4.31) should be understood as the system of four scalar equations:

$$\chi_{lk}(\lambda) = \delta_{lk} + 2t\int_{\gamma_k(\lambda)} \mathrm{e}^{2t\int_{\zeta}^{\lambda}(\Lambda_{ll}(z)-\Lambda_{kk}(z))\,\mathrm{d}z}(R(\zeta)\chi(\zeta))_{lk}\,\mathrm{d}\zeta.$$

Let $\gamma_{\pm}^{1,3}$ be the anti-Stokes' lines defined by the equations,

$$\operatorname{Im}\int_{\lambda_{1,3}}^{\lambda} \mu(z)\,\mathrm{d}z = 0$$

and asymptotic to the the rays, $\arg\lambda = \pm\frac{\pi}{4}$, $(\gamma_{\pm}^{1})$ and $\arg\lambda = \pm\frac{\pi}{2}$, $(\gamma_{\pm}^{3})$. We denote $\mathcal{D}_{\pm}$ the corresponding canonical domains:

$$\partial\mathcal{D}_{\pm} = \gamma_{\pm}^{1} \cup \gamma_{\pm}^{3} \cup [\lambda_3, \lambda_1]. \tag{4.33}$$

We note that domain $\mathcal{D}_{+}(\mathcal{D}_{-})$ contains exactly one Stokes ray, i.e. the ray, $\arg\lambda = \frac{3\pi}{8}(-\frac{3\pi}{8})$.

We shall use universal symbol $\mathcal{D}$ for the canonical domains when the distinction between $\mathcal{D}_{+}$ and $\mathcal{D}_{-}$ does not play any role.

Being a canonical domain means exactly (cf [26]) that for each $\lambda \in \mathcal{D}$ there exists matrix $\gamma(\lambda)$ of the indicated above canonical paths such that:

(1) $\gamma_{1,2}(\lambda) \subset \mathcal{D} \qquad \forall\lambda \in \mathcal{D}$,

(2) for any two points, $\lambda, \lambda' \in \mathcal{D}$, the following equation takes place:

$$\gamma_j(\lambda) - \gamma_j(\lambda') + [\lambda', \lambda] = \partial\Omega_j(\lambda, \lambda') \qquad j = 1, 2$$

for some bounded $\Omega_j(\lambda, \lambda') \subset \mathcal{D}$. In other words, any two $\gamma_j(\lambda), \gamma_j(\lambda')$ have the same infinite parts.

Integral equation (4.31) is written for the canonical domain $\mathcal{D}$. Property (4.32) of the canonical paths and properties (1) and (2) of the canonical domain imply that the integral operator on the r.h.s. of (4.31) is well defined as a bounded operator on the Banach space of holomorphic and bounded in $\mathcal{D}$ matrix functions. Our next task is to estimate the norm of this integral operator.

To analyse integral equation (4.31) we need some basic estimates. Let

$$\begin{aligned} &\mathcal{D}^{\delta} = \{\lambda \in \mathcal{D} : \operatorname{dist}\{\lambda; \partial\mathcal{D}\} \geqslant t^{-\frac{1}{2}+\delta}\} \\ &\tfrac{1}{2} > \delta > 0 \end{aligned} \tag{4.34}$$

and rewrite the entries a_3, a_+ in the form:

$$\begin{aligned} &a_3 = \frac{1}{2}\lambda\left(\lambda^2 - \frac{2}{3}\right)\left(1 + \frac{g}{t^{1/2}(\lambda^2 - \frac{2}{3})}\right) \\ &g = 2t^{1/2}\left(r - \frac{2}{3}\right) + \frac{\alpha}{t^{1/2}\lambda^2} \qquad |g| \leqslant C \qquad t \geqslant 1 \qquad |\lambda| \geqslant \rho > 0 \end{aligned} \tag{4.35}$$

$$\begin{aligned} &a_+ = \mathrm{i}r\left(\lambda^2 - \frac{2}{3}\right)\left(1 + \frac{h}{t^{1/2}(\lambda^2 - \frac{2}{3})}\right) \\ &h = \frac{t^{1/2}}{2}\left(r - \frac{2}{3}\right) + \frac{\beta}{4t^{1/2}r} - \frac{t^{1/2}}{2r}\left(r' + \frac{r}{2t}\right) \qquad |h| \leqslant C \qquad t \geqslant t_0. \end{aligned} \tag{4.36}$$

Let us also expand the functions $\mu(\lambda)$, $\mu^{-1}(\lambda)$ and a_+^{-1} in the series over the inverse powers of $\lambda^2 - \frac{2}{3}$:

$$\mu^{\pm 1}(\lambda) = \left[\frac{1}{2}\left(\lambda^2 - \frac{2}{3}\right)\sqrt{\lambda^2 - \frac{8}{3}}\right]^{\pm 1}\left(1 \pm \sum_{n=1}^{\infty} c_n^{\pm}\frac{\nu^n(\lambda)}{t^n(\lambda^2 - \frac{2}{3})^{2n}}\right) \tag{4.37}$$

$$\frac{1}{a_+} = \frac{-\mathrm{i}}{r(\lambda^2 - \frac{2}{3})}\left(1 + \sum_{n=1}^{\infty}(-1)^n\frac{h^n}{t^{n/2}(\lambda^2 - \frac{2}{3})^n}\right) \qquad -\mathrm{i} = \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}} \tag{4.38}$$

where

$$c_n^- = (2n-1)c_n^+ = \frac{(-1)^{n-1}}{2^n}\frac{(2n-1)!!}{n!}$$
$$\nu(\lambda) = 4\left(\lambda^2 - \frac{8}{3}\right)^{-1}\left(\frac{\alpha - \beta}{2}\lambda^2 + tp + \frac{\alpha^2}{4t\lambda^2}\right) \tag{4.39}$$

and h is given in (4.33). From (4.39) and (4.16) we immediately conclude that

$$|\nu(\lambda)| \leqslant C \qquad |\nu'(\lambda)| \leqslant \frac{C}{|\lambda|^3} \qquad |\nu''(\lambda)| \leqslant \frac{C}{|\lambda|^4}$$
$$\forall\lambda \in \mathcal{D}^{\delta} \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_0.$$

These inequalities, together with the similar inequalities for g and h (see (4.35) and (4.36)), imply that there exists positive constant $t_1 = t_1(s_-^*; \varepsilon; \delta)$ such that the series in (4.37) and (4.38) and their term-by-term derivatives with respect to λ converge uniformly for

$$(\lambda, s_-, t) \in \mathcal{D}^{\delta} \times D(s_-^*; \varepsilon) \times [t_1, +\infty).$$

This fact justifies the obvious sequence of elementary formal manipulations with the series (4.37), (4.38) which leads to the following uniform estmates for the quantaties involved in the matrices $H_{1,2}$:

$$\left|\frac{a_+}{2\mu}\right| \leqslant C \tag{4.40}$$

$$\frac{1}{|1 + \frac{a_+}{4t\mu^2}(\frac{a_3}{a_+})'|} \leqslant C \tag{4.41}$$

$$\left|\frac{1}{2\mu}\left(\frac{\mu \pm a_3}{a_+}\right)'\right| \leqslant C\frac{|\lambda|}{|\lambda^2 - \frac{2}{3}|^2} \tag{4.42}$$

$$|a_+|\left|\frac{1}{2\mu}\left(\frac{\mu \pm a_3}{a_+}\right)'\right|^2 \leqslant C\frac{|\lambda|^2}{|\lambda^2 - \frac{2}{3}|^3} \tag{4.43}$$

$$\left|\left(\frac{1}{2\mu}\left(\frac{\mu \pm a_3}{a_+}\right)'\right)'\right| \leqslant C\frac{|\lambda|^2}{|\lambda^2 - \frac{2}{3}|^3} \tag{4.44}$$

$$\forall\lambda \in \mathcal{D}^{\delta} \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_1.$$

We emphasize (cf remark 4.1) that positive constants C and t_0 in all our formulae depend only on s_-^* and ε:

$$C, t_0 = C(s_-^*; \varepsilon), t_0(s_-^*; \varepsilon).$$

Positive constant t_1 depends only on s_-^*, ε and δ:

$$t_1 = t_1(s_-^*; \varepsilon; \delta).$$

Inequalities (4.40)–(4.44) imply the following proposition.

Proposition 4.1. For every $s_- \in D(s_-^*; \varepsilon)$ and $t > t_1$ reminder $R(\lambda)$ defined in (4.29) is holomorphic in $\mathcal{D}^\delta$ and satisfies the uniform estimate

$$\|R(\lambda)\| \leqslant C \frac{|\lambda|^2}{t^2|\lambda^2 - \frac{2}{3}|^3} \qquad \forall\lambda \in \mathcal{D}^\delta \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_1. \tag{4.45}$$

To estimate the part,

$$\exp\left\{\int_\zeta^\lambda \eta(z)\,\mathrm{d}z\right\}$$

of the kernel of integral equation (4.31) we note that from (4.35), (4.36), and (4.38) it follows that

$$\left|a_3' - a_3\frac{a_+'}{a_+} - \frac{1}{2}\left(\lambda^2 - \frac{2}{3}\right)\right| \leqslant C\frac{|\lambda|^2}{\sqrt{t}|\lambda^2 - \frac{2}{3}|} \qquad \forall\lambda \in \mathcal{D}^\delta \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_1. \tag{4.46}$$

Simultaneously, series (4.37) yields the equation,

$$\frac{1}{\mu(\lambda)} = \frac{2}{\sqrt{\lambda^2 - \frac{8}{3}}(\lambda^2 - \frac{2}{3})}(1 + \mu_0) \tag{4.47}$$

where

$$|\mu_0| \leqslant \frac{C}{t|\lambda^2 - \frac{2}{3}|^2} \leqslant \frac{C_1}{t^{2\delta}}. \tag{4.48}$$

for all

$$\lambda \in \mathcal{D}^\delta \qquad s_- \in D(s_-^*; \varepsilon) \qquad \text{and} \qquad t \geqslant t_1.$$

Inequalities (4.46)–(4.48) lead to the estimates,

$$\left|\frac{1}{2\mu}\left(a_3' - a_3\frac{a_+'}{a_+} - \frac{1}{2}\left(\lambda^2 - \frac{2}{3}\right)\right)\right| \leqslant \frac{C}{\sqrt{t}}\frac{|\lambda|}{|\lambda^2 - \frac{2}{3}|^2} \tag{4.49}$$

and

$$\begin{aligned}
&\left|\int_\zeta^\lambda \left(a_3'(z) - a_3(z)\frac{a_+'(z)}{a_+(z)} - \frac{1}{2}\left(z^2 - \frac{2}{3}\right)\right)\frac{\mathrm{d}z}{2\mu(z)}\right| \\
&\quad = \left|\left\{\int_\zeta^\infty + \int_\infty^\lambda\right\}\left(a_3'(z) - a_3(z)\frac{a_+'(z)}{a_+(z)} - \frac{1}{2}\left(z^2 - \frac{2}{3}\right)\right)\frac{\mathrm{d}z}{2\mu(z)}\right| \\
&\quad \leqslant \frac{C}{\sqrt{t}}\int_\zeta^\infty \frac{|z|}{|z^2 - \frac{2}{3}|^2}|\mathrm{d}z| + \frac{C}{\sqrt{t}}\int_\lambda^\infty \frac{|z|}{|z^2 - \frac{2}{3}|^2}|\mathrm{d}z| \leqslant \frac{C_1}{t^\delta} \\
&\quad \forall\lambda, \zeta \in \mathcal{D}^\delta \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_1
\end{aligned} \tag{4.50}$$

(the integration in the last two integrals is performing along the rays in $\mathcal{D}^\delta$). This in turn means that

$$\left|\exp\left\{\int_\zeta^\lambda \left(a_3'(z) - a_3(z)\frac{a_+'(z)}{a_+(z)} - \frac{1}{2}\left(z^2 - \frac{2}{3}\right)\right)\frac{\mathrm{d}z}{2\mu(z)}\right\} - 1\right| \leqslant \frac{C}{t^\delta} \qquad \forall\lambda, \zeta \in \mathcal{D}^\delta \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_1. \tag{4.51}$$

At the same time, because of (4.47), (4.48) we have

$$\int_{\zeta}^{\lambda} \frac{z^2 - \frac{2}{3}}{4\mu(z)}\,\mathrm{d}z = \frac{1}{2}\ln\frac{\lambda + \sqrt{\lambda^2 - \frac{8}{3}}}{\zeta + \sqrt{\zeta^2 - \frac{8}{3}}} + \mu_1 \tag{4.52}$$

where

$$|\mu_1| \leqslant \frac{C}{t}\int_{\zeta}^{\infty} \frac{|\mathrm{d}z|}{\left|\sqrt{z^2 - \frac{8}{3}}\right|\left|z^2 - \frac{2}{3}\right|^2} + \frac{C}{t}\int_{\lambda}^{\infty} \frac{|\mathrm{d}z|}{\left|\sqrt{z^2 - \frac{8}{3}}\right|\left|z^2 - \frac{2}{3}\right|^2} \leqslant \frac{C_1}{t^{\frac{1}{2}+\delta}}$$
$$\forall\lambda, \zeta \in \mathcal{D}^{\delta} \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_1. \tag{4.53}$$

Formulae (4.51)–(4.53) imply the following proposition.

Proposition 4.2. The function (see (4.31)),

$$\mathrm{e}^{\int_{\zeta}^{\lambda}\eta(z)\,\mathrm{d}z}$$

satisfies the uniform estimate,

$$|\mathrm{e}^{\int_{\zeta}^{\lambda}\eta(z)\,\mathrm{d}z}| \leqslant C\left|\frac{\lambda}{\zeta}\right|^{1/2}$$
$$\forall\lambda, \zeta \in \mathcal{D}^{\delta} \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_1. \tag{4.54}$$

Let K denote the integral operator on the r.h.s. of equation (4.31). In virtue of the property (4.32) of the canonical path and estimates (4.45), (4.54), the $C(\mathcal{D}^{\delta})$-norm of K satisfies the inequality,

$$\|K\|_{C(\mathcal{D}^{\delta})} \leqslant C\frac{|\lambda|}{t|\lambda^2 - \frac{2}{3}|^2} \leqslant \frac{1}{2}$$
$$\forall\lambda \in \mathcal{D}^{\delta} \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_1. \tag{4.55}$$

This means that integral equation (4.31) is uniquely solvable in the Banach space of holomorphic and bounded in $\mathcal{D}^{\delta}$ functions and that its solution $\chi(\lambda)$ satisfies the uniform estimate

$$\|\chi(\lambda) - I\| \leqslant C\frac{|\lambda|}{t|\lambda^2 - \frac{2}{3}|^2} \leqslant \frac{C_1}{t^{2\delta}|\lambda|^3}.$$
$$\forall\lambda \in \mathcal{D}^{\delta} \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad \forall t > t_1. \tag{4.56}$$

Along with the matrix $T(\lambda)$ (see (4.25)), let us consider the matrix

$$T_0 = \begin{pmatrix} 1 & 1 \\ h_1^0 & h_2^0 \end{pmatrix} \tag{4.57}$$

where

$$h_1^0 = \frac{\mu - a_3}{a_+} \qquad h_2^0 = -\frac{\mu + a_3}{a_+}.$$

Since

$$T_0^{-1}(\lambda)T(\lambda) = \begin{pmatrix} 1 - \frac{1}{2t}\cdot\frac{a_+}{4\mu^2}\left(\frac{\mu - a_3}{a_+}\right)' & -\frac{1}{2t}\cdot\frac{a_+}{4\mu^2}\left(\frac{\mu + a_3}{a_+}\right)' \\ \frac{1}{2t}\cdot\frac{a_+}{4\mu^2}\left(\frac{\mu - a_3}{a_+}\right)' & 1 + \frac{1}{2t}\cdot\frac{a_+}{4\mu^2}\left(\frac{\mu + a_3}{a_+}\right)' \end{pmatrix}$$

we can use inequalities (4.40), (4.42) again and conclude that

$$\|T_0^{-1}(\lambda)T(\lambda) - I\| \leqslant C\frac{|\lambda|}{t|\lambda^2 - \frac{2}{3}|^2} \leqslant \frac{C_1}{t^{2\delta}|\lambda|^3} \qquad \forall\lambda \in \mathcal{D}^\delta \qquad \forall s_- \in D(s_-^*;\varepsilon) \qquad \forall t > t_1. \tag{4.58}$$

Equations (4.24) and (4.30) together with estimates (4.56) and (4.58) lead to the following basic lemma.

WKB lemma. Let $\mathcal{D}_\pm^\delta$ be the canonical domains defined by equations (4.33) and (4.34). Then in each region, $\mathcal{D}_\pm$, there exists a WKB-solution, $\Phi_\pm^{\mathrm{WKB}}(\lambda)$ of system (4.13), which admits the following representation:

$$\Phi_\pm^{\mathrm{WKB}}(\lambda) = T_0(\lambda)Y_\pm^{\mathrm{WKB}}(\lambda) = T_0(\lambda)\chi_\pm(\lambda)\mathrm{e}^{2t\int_{\lambda_0}^{\lambda_\pm}\Lambda(z)\,\mathrm{d}z}. \tag{4.59}$$

The matrix functions $T_0(\lambda)$ and $\Lambda(\lambda)$ are given by explicit formulae (4.57) and (4.28) respectively. Matrix functions $\chi_+(\lambda)$ and $\chi_-(\lambda)$ are holomorphic in $\mathcal{D}_+^\delta$ and $\mathcal{D}_-^\delta$ respectively and satisfy the uniform estimates,

$$\|\chi_\pm(\lambda) - I\| \leqslant \frac{C}{t^{2\delta}|\lambda|^3} \qquad \forall\lambda \in \mathcal{D}_\pm^\delta \qquad \forall s_- \in D(s_-^*;\varepsilon) \qquad \forall t > t_1. \tag{4.60}$$

In (4.60), positive constant C depends only on s_-^* and ε:

$$C = C(s_-^*;\varepsilon)$$

positive constant t_1 depends only on s_-^*, ε, and δ:

$$t_1 = t_1(s_-^*;\varepsilon;\delta).$$

Step 2. The local solution near the double turning point. We consider the neighbourhood of the double turning point $\sqrt{\frac{2}{3}}$:

$$U = \left\{\lambda : \left|\lambda - \sqrt{\tfrac{2}{3}}\right| \leqslant 2t^{-\frac{1}{2}+\delta}\right\} \qquad 0 < \delta < \tfrac{1}{2}. \tag{4.61}$$

To construct the local asymptotic solution in the area (4.61) it is convenient to make the gauge transformation

$$\Phi(\lambda) = VZ(\lambda) \qquad V = \begin{pmatrix} 1 & 1 \\ \sqrt{\frac{3}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} & -\sqrt{\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}} \end{pmatrix}. \tag{4.62}$$

The function $Z(\lambda)$ satisfies the equation

$$\frac{\partial Z}{\partial\lambda} = 2t\hat{A}Z \qquad \hat{A} = \hat{a}_3\sigma_3 + \hat{a}_+\sigma_+ + \hat{a}_-\sigma_- \tag{4.63}$$

where

$$\begin{aligned} \hat{a}_3 &= \frac{1}{\sqrt{2}}\left\{-\mathrm{i}\sqrt{\frac{2}{3}}a_3 + a_+ + \frac{2}{3}a_-\right\} \\ \hat{a}_+ &= \mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}\frac{1}{\sqrt{2}}\left\{2\sqrt{\frac{2}{3}}a_3 - \mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}a_+ + \frac{2}{3}\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}a_-\right\} \\ \hat{a}_- &= \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\frac{1}{\sqrt{2}}\left\{2\sqrt{\frac{2}{3}}a_3 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}a_+ - \frac{2}{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}a_-\right\} \end{aligned} \tag{4.64}$$

and we recall (cf (4.13)) that

$$a_3 = \frac{1}{2}\lambda^3 + \lambda(r-1) + \frac{\alpha}{2t\lambda}$$
$$a_+ = \mathrm{i}r\left(\lambda^2 - 1 + \frac{r}{2} + \frac{\beta}{4tr} - \left(\frac{r'}{2r} + \frac{1}{4t}\right)\right)$$
$$a_- = \mathrm{i}\left(\lambda^2 - 1 + \frac{r}{2} + \frac{\beta}{4tr} + \left(\frac{r'}{2r} + \frac{1}{4t}\right)\right).$$

In the area (4.61), due to (4.15), the matrix (4.63) can be represented in the form

$$\hat{A}(\lambda) = B_0(\lambda) + R_0(\lambda) \tag{4.65}$$

where

$$B_0(\lambda) = \mathrm{i}\frac{2}{\sqrt{3}}\left(\lambda - \sqrt{\frac{2}{3}}\right)\sigma_3 + a\sqrt{\frac{2}{t}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{6} - \mathrm{i}\Theta}\sigma_+ \\ + a\sqrt{\frac{2}{t}}\mathrm{e}^{\mathrm{i}\frac{\pi}{6} + \mathrm{i}\Theta}\sigma_- \equiv b_3\sigma_3 + b_+\sigma_+ + b_-\sigma_- \qquad b_- = \bar{b}_+ \tag{4.66}$$

while for the matrix $R_0(\lambda)$ the following inequality holds:

$$\begin{gathered} \|R_0(\lambda)\| \leqslant t^{-1+2\delta}C \\ \forall s \in D(s_-^*; \varepsilon) \qquad \forall t > t_0(s_-^*; \varepsilon) \\ \forall \lambda : \left|\lambda - \sqrt{\tfrac{2}{3}}\right| \leqslant 2t^{-\frac{1}{2}+\delta} \qquad 0 < \delta < \tfrac{1}{2}. \end{gathered} \tag{4.67}$$

The model equation

$$\frac{\mathrm{d}Z_0}{\mathrm{d}\lambda} = 2tB_0Z_0 \tag{4.68}$$

is exactly solvable in terms of the Weber–Hermite functions $D_\nu(z)$ (see [27]):

$$Z_0(\lambda) = \begin{pmatrix} D_{-\nu-1}(\mathrm{i}z) & D_\nu(z) \\ \dot{D}_{-\nu-1}(\mathrm{i}z) & \dot{D}_\nu(z) \end{pmatrix} \tag{4.69}$$

where

$$\begin{gathered} z = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\frac{2\sqrt{2t}}{\sqrt[4]{3}}\left(\lambda - \sqrt{\frac{2}{3}}\right) \\ \nu + 1 = \mathrm{i}\frac{\sqrt{3}}{2}tb_+b_- = \mathrm{i}\sqrt{3}a^2 \end{gathered} \tag{4.70}$$

$$\dot{D} = \frac{1}{b_0}\left(\frac{\mathrm{d}D}{\mathrm{d}z} - \frac{z}{2}D\right) \qquad b_0 = \sqrt[4]{3}a\mathrm{e}^{-\mathrm{i}\frac{5\pi}{12} - \mathrm{i}\Theta}. \tag{4.71}$$

The local solution of equation (4.63) can be defined now as the product (cf (4.30)),

$$Z(\lambda) = \chi_0(\lambda)Z_0(\lambda) \tag{4.72}$$

where the matrix function $\chi_0(\lambda)$ satifies the integral equation

$$\chi_0(\lambda) = I + 2t\int_{\sqrt{2/3}}^{\lambda} Z_0(\lambda)Z_0^{-1}(\zeta)R(\zeta)\chi_0(\zeta)Z_0(\zeta)Z_0^{-1}(\lambda)\,\mathrm{d}\zeta. \tag{4.73}$$

This is a Volterra equation with the regular kernel. As it follows from the known (see e.g. [27]) integral representations and asymptotic expansions for the parabolic cylinder functions, the functions $D_{-\nu-1}(z(\lambda))$, $D_\nu(z(\lambda))$, and their derivatives are bounded uniformly with respect to $s_- \in D(s_-^*;\varepsilon)$, if

$$\lambda \in \mathcal{S} = \{\lambda \in U : |\mathrm{Re}\, z^2(\lambda)| \leqslant 1\}. \tag{4.74}$$

This yields immediately the following estimate in the star-shaped region defined in (4.74):

$$\|Z_0(\lambda)Z_0^{-1}(\zeta)\| \leqslant C \qquad \forall s_- \in D(s_-^*;\varepsilon) \qquad z,\zeta \in \mathcal{S}. \tag{4.75}$$

The Volterra equation (4.73) has a unique solution $\chi_0(\lambda)$, which is analytic in the whole neighbourhood U and satisfies there the estimate,

$$\|\chi_0(\lambda) - I\| \leqslant \mathrm{e}^{\sigma(\lambda)} - 1 \tag{4.76}$$

where

$$\sigma(\lambda) = 2t\int_{\sqrt{2/3}}^{\lambda} \|Z_0(\lambda)Z_0^{-1}(\zeta)\| \cdot \|R(\zeta)\| \cdot \|Z_0(\zeta)Z_0^{-1}(\lambda)\| |\mathrm{d}\zeta|$$

and the integration is performing along the radius of the disk U. Because of (4.67) and (4.75), in the star-shaped region $\mathcal{S}$ the inequality,

$$\sigma(\lambda) \leqslant 2t^{-\frac{1}{2}+3\delta}C$$

holds. This leads to the following lemma.

Turning point lemma. Let U and S be the disk and the star-shaped region defined by equations (4.61) and (4.74), respectively. Then in the disk U, there exists a turning point solution, $\Phi^{\mathrm{TP}}(\lambda)$, of system (4.13), which admits the following representation:

$$\Phi^{\mathrm{TP}}(\lambda) = VZ(\lambda) = V\chi_0(\lambda)Z_0(\lambda). \tag{4.77}$$

Matrix V and the matrix function $Z_0(\lambda)$ are given by explicit formulae (4.62) and (4.69), respectively. Matrix function $\chi_0(\lambda)$ is holomorphic in U and satisfies the uniform estimate,

$$\begin{gathered}\|\chi_0(\lambda) - I\| \leqslant t^{-\frac{1}{2}+3\delta}C \qquad 0 < \delta < \tfrac{1}{6} \\ \forall s \in D(s_-^*;\varepsilon) \qquad \forall\lambda \in \mathcal{S} \qquad t > t_0\end{gathered} \tag{4.78}$$

in the star-shaped region S. In (4.78), the positive constants C and t_0 depend only on s_-^* and ε:

$$C = C(s_-^*;\varepsilon) \qquad t_0 = t(s_-^*;\varepsilon).$$

Step 3. Calculation of the matrices $C_\pm$. Let us consider the matrix

$$C_+ = [\Phi_+^{\mathrm{WKB}}(\lambda)]^{-1}\Psi_2(\xi(\lambda)) = [T_0(\lambda)Y_+^{\mathrm{WKB}}(\lambda)]^{-1}\Psi_2(\xi(\lambda)). \tag{4.79}$$

The matrices $C_\pm$ *a priori* do not depend on λ. In particular, this means that when evaluating C_+ we can use the equation,

$$C_+ = \lim_{\substack{\lambda\to\infty\\ \lambda\in\mathcal{D}_+}} \{[T_0(\lambda)Y_+^{\mathrm{WKB}}(\lambda)]^{-1}\Psi_2(\xi(\lambda))\}. \tag{4.80}$$

Taking into account the fact that in the domain $\mathcal{D}_+$,

$$\mu(\lambda) \sim \tfrac{1}{2}\lambda^3 - \lambda \qquad \lambda \to \infty \tag{4.81}$$

we derive from (4.57) the asymptotic equation,

$$T_0^{-1}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{\mathrm{i}r}{\lambda} \end{pmatrix} \left\{ I + \mathcal{O}\left(\frac{1}{\lambda}\right) \right\} \qquad \lambda \to \infty \qquad \lambda \in \mathcal{D}_+. \tag{4.82}$$

Equation (4.82) together with the equation,

$$Y_+^{\mathrm{WKB}}(\lambda) = \chi_+(\lambda)\mathrm{e}^{2t\int_{\lambda_0^+}^{\lambda} \Lambda(z)\,\mathrm{d}z}$$

and estimate (4.60) imply the following explicit formula for the matrix C_+:

$$C_+ = \sqrt{\frac{a_+}{\mu}}\bigg|_{\lambda_0^+} \lim_{\substack{\lambda\to\infty \\ \lambda\in\mathcal{D}_+}} \left\{ \sqrt{\frac{\mu}{a_+}}\bigg|_{\lambda} \exp\left(-2t\int_{\lambda_0^+}^{\lambda} \Lambda_3\,\mathrm{d}\lambda\,\sigma_3 + \left(\frac{1}{8}\xi^4 + \frac{x}{2}\xi^2 + (\alpha-\beta)\ln\xi\right)\sigma_3 \right) \times \begin{pmatrix} 1 & \\ & -\frac{\mathrm{i}r}{\lambda} \end{pmatrix} \right\} \tag{4.83}$$

where the notation Λ_3 is used for the σ_3-component of the diagonal matrix Λ (see 4.28):

$$\Lambda_3 = \mu + \frac{1}{4t\mu}\left(a_3' - a_3\frac{a_+'}{a_+}\right).$$

Similarly,

$$C_- = [\Phi_-^{\mathrm{WKB}}(\lambda)]^{-1}\Psi_{-1}(\xi(\lambda)) \tag{4.84}$$

$$= \sqrt{\frac{a_+}{\mu}}\bigg|_{\lambda_0^-} \lim_{\substack{\lambda\to\infty \\ \lambda\in\mathcal{D}_-}} \left\{ \sqrt{\frac{\mu}{a_+}}\bigg|_{\lambda} \exp\left(-2t\int_{\lambda_0^-}^{\lambda} \Lambda_3\,\mathrm{d}\lambda\,\sigma_3 - \left(\frac{1}{8}\xi^4 + \frac{x}{2}\xi^2 + (\alpha-\beta)\ln\xi\right)\sigma_3 \right) \begin{pmatrix} -\frac{\mathrm{i}r}{\lambda} & \\ & 1 \end{pmatrix} \right\}\sigma_1 \tag{4.85}$$

where we have taken into account the fact that in the domain $\mathcal{D}_-$ estimates (4.81) and (4.82) should be replaced by the estimates

$$\mu(\lambda) \sim -\tfrac{1}{2}\lambda^3 + \lambda \qquad \xi \to \infty \tag{4.86}$$

and

$$T_0^{-1}(\lambda) = \begin{pmatrix} -\frac{\mathrm{i}r}{\lambda} & 0 \\ 0 & 1 \end{pmatrix}\sigma_1 \left\{ I + \mathcal{O}\left(\frac{1}{\lambda}\right) \right\} \tag{4.87}$$

respectively.

Equations (4.83) and (4.84) in turn can be used for evaluation the asymptotics of the matrices $C_\pm$ as $t \to \infty$. Indeed,

$$2t\int_{\lambda_0}^{\lambda} \Lambda_3\,\mathrm{d}\lambda = 2t(g(\lambda) - g(\lambda_0)) + I(\lambda, \lambda_0) \tag{4.88}$$

where the function $g(\lambda)$ is given by the equation,

$$g(\lambda) = \frac{1}{8}\lambda\left(\lambda^2 - \frac{8}{3}\right)^{3/2} + \frac{1}{2t}\left\{ \left(\alpha - \beta + \frac{1}{2}\right)\ln\left(\lambda + \sqrt{\lambda^2 - \frac{8}{3}}\right) - \mathrm{i}\sqrt{3}a^2 \ln\left[\frac{\left(\lambda + \sqrt{\lambda^2 - \frac{8}{3}}\right)^2 - \frac{2}{3}(1+\mathrm{i}\sqrt{3})^2}{\left(\lambda + \sqrt{\lambda^2 - \frac{8}{3}}\right)^2 - \frac{2}{3}(1-\mathrm{i}\sqrt{3})^2} \right] \right\} \tag{4.89}$$

and the remainder $I(\lambda,\lambda_0)$ satisfies the following uniform estimate:

$$
\begin{aligned}
&|I(\lambda,\lambda_0)| \leqslant t^{-\delta}C \\
&\forall(\lambda,\lambda_0)\in\mathcal{D}^\delta\times\mathcal{D}^\delta \qquad \forall s_-\in D(s_-^*;\varepsilon) \qquad \forall t>t_1.
\end{aligned}
\tag{4.90}
$$

In (4.89), the function $\sqrt{\lambda^2-\frac{8}{3}}$ is understood as a single-valued analytic function on $\mathbb{C}\backslash(-\infty,-\sqrt{\frac{8}{3}}]\cup[\sqrt{\frac{8}{3}},+\infty)$ which has the asymptotics, $\sqrt{\lambda^2-\frac{8}{3}}\sim\lambda$, as $\lambda\to\mathrm{i}\infty$ (cf with the definition of $\mu(\lambda)$). The branches of the rest of the involved multivalued functions are fixed by the conditions,

$$
0<\arg\left[\lambda+\sqrt{\lambda^2-\tfrac{8}{3}}\right]<\pi \qquad \lambda\in\mathcal{D}_\pm \tag{4.91}
$$

$$
0<\arg\left[\left(\lambda+\sqrt{\lambda^2-\tfrac{8}{3}}\right)^2-\tfrac{2}{3}(1+\mathrm{i}\sqrt{3})^2\right]<\pi \qquad \lambda\in\mathcal{D}_+^\delta \tag{4.92}
$$

$$
-\pi<\arg\left[\left(\lambda+\sqrt{\lambda^2-\tfrac{8}{3}}\right)^2-\tfrac{2}{3}(1+\mathrm{i}\sqrt{3})^2\right]<0 \qquad \lambda\in\mathcal{D}_-^\delta \tag{4.93}
$$

$$
0<\arg\left[\left(\lambda+\sqrt{\lambda^2-\tfrac{8}{3}}\right)^2-\tfrac{2}{3}(1-\mathrm{i}\sqrt{3})^2\right]<\pi \qquad \lambda\in\mathcal{D}_\pm^\delta. \tag{4.94}
$$

From equation (4.89) and conditions (4.91)–(4.94) it follows that

$$
g(\lambda)=\frac{\lambda^4}{8}-\frac{\lambda}{2}+\frac{1}{3}+\frac{1}{2t}\left(\alpha-\beta+\frac{1}{2}\right)\ln 2\lambda+\mathcal{O}(\lambda^{-2}) \qquad 0<\arg\lambda<\frac{\pi}{2} \tag{4.95}
$$

as $\lambda\to\infty \qquad \lambda\in\mathcal{D}_+$, and

$$
\begin{aligned}
g(\lambda)=&-\frac{\lambda^4}{8}+\frac{\lambda}{2}-\frac{1}{3}-\frac{1}{2t}\left(\alpha-\beta+\frac{1}{2}\right)\ln\frac{3\lambda}{4} \\
&-\frac{\pi}{t\sqrt{3}}a^2+\mathcal{O}(\lambda^{-2}) \qquad -\frac{\pi}{2}<\arg\lambda<0
\end{aligned}
\tag{4.96}
$$

as $\lambda\to\infty$, $\lambda\in\mathcal{D}_-$. Taking into account also the uniform with respect to $s_-\in D(s_-^*;\varepsilon)$ and $t>t_1(s_-^*;\varepsilon;\delta)$ estimates (see (4.37), (4.38)),

$$
\sqrt{\frac{\mu}{a_+}}=\frac{\sqrt{3\lambda}}{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\left[1+\mathcal{O}\left(\frac{1}{\lambda^2}+\frac{1}{t^\delta}\right)\right] \qquad 0<\arg\lambda<\frac{\pi}{2} \tag{4.97}
$$

as $\lambda\to\infty$, $\lambda\in\mathcal{D}_+^\delta$, and

$$
\sqrt{\frac{\mu}{a_+}}=\frac{\sqrt{3\lambda}}{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\left[1+\mathcal{O}\left(\frac{1}{\lambda^2}+\frac{1}{t^\delta}\right)\right] \qquad -\frac{\pi}{2}<\arg\lambda<0 \tag{4.98}
$$

as $\lambda\to\infty$, $\lambda\in\mathcal{D}_-^\delta$, we end up with the following asymptotic equations for the matrices $C_\pm$:

$$
C_+=\sqrt{\frac{a_+}{\mu}}\bigg|_{\lambda_0^+}\{I+c_+(t)\}\mathrm{e}^{(2tg(\lambda_0^+)-\frac{2t}{3}+\frac{\alpha-\beta}{4}\ln 2t+\omega_+)\sigma_3}\frac{\sigma_3}{\sqrt{2}} \tag{4.99}
$$

$$
C_-=\sqrt{\frac{a_+}{\mu}}\bigg|_{\lambda_0^-}\{I+c_-(t)\}\mathrm{e}^{(2tg(\lambda_0^-)+\frac{2t}{3}-\frac{\alpha-\beta}{4}\ln 2t+\omega_-)\sigma_3}\frac{\sigma_1}{\sqrt{2}} \tag{4.100}
$$

where

$$\begin{aligned}\omega_+ &= -\frac{i\pi}{4} - \frac{1}{2}\ln\frac{2}{3} - \left(\alpha - \beta + \frac{1}{2}\right)\ln 2\\ \omega_- &= -\frac{i\pi}{4} + \frac{1}{2}\ln\frac{2}{3} + \left(\alpha - \beta + \frac{1}{2}\right)\ln\frac{3}{4} + \frac{2\pi}{\sqrt{3}}a^2\end{aligned} \tag{4.101}$$

and the matrix functions $c_\pm(t)$ are diagonal and satisfy the uniform estimate,

$$|c_+| < \frac{C}{t^\delta}$$
$$t \geqslant t_1 \qquad s \in D(s_-^*; \epsilon) \qquad \lambda_0^\pm \in \mathcal{D}_\pm^\delta.$$

Step 4. Calculation of the matrices $N_\pm$. The connection matrices $N_\pm$ are defined by equations (cf (4.20)),

$$N_\pm = [\Phi^{\mathrm{TP}}(\lambda)]^{-1}\Phi_\pm^{\mathrm{WKB}}(\lambda) = \sqrt{\frac{\mu}{a_+}}\bigg|_{\lambda_0^\pm}\sqrt{\frac{a_+}{\mu}}\bigg|_\lambda Z_0^{-1}(\lambda)\chi_0^{-1}(\lambda)V^{-1}T_0(\lambda)\chi_\pm(\lambda)\mathrm{e}^{2t\int_{\lambda_0^\pm}^{\lambda}\Lambda_3\,\mathrm{d}\lambda\sigma_3}. \tag{4.102}$$

Similar to $C_\pm$, the matrices $N_\pm$ do not depend on λ. This means that when evaluating $N_\pm$ we may assume that

$$\lambda \in \mathcal{P}_\pm \equiv \mathcal{D}_\pm^\delta \cap \mathcal{S} \qquad 0 < \delta < \tfrac{1}{6}. \tag{4.103}$$

The obvious advantage of this choice is that in the matching area (4.103) both the functions, $\chi_0(\lambda)$ and $\chi_\pm(\lambda)$, are asymptotically close to the unit matrix as $t \to \infty$.

It follows from the definitions of the matrices V and $T_0(\lambda)$ (see (4.62) and (4.57)) and from estimates (4.37) and (4.38) that

$$\|V^{-1}T_0(\lambda) - I\| \leqslant t^{-\delta}C \qquad \forall\lambda \in \mathcal{P}_\pm \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad t > t_1. \tag{4.104}$$

On the other hand, if z is the variable defined in (4.70), then in the regions (4.103) we have that

$$|z| \to \infty \qquad \text{and} \qquad \arg z \to \begin{cases} \dfrac{3\pi}{4} & \text{for } \lambda \in \mathcal{P}_+\\ -\dfrac{\pi}{4} & \text{for } \lambda \in \mathcal{P}_-\end{cases} \tag{4.105}$$

as $t \to \infty$. Therefore, in (4.69) we can use the known large z asymptotic expansions of the parabolic cylinder functions (see e.g. [27]). This yields the equation,

$$Z_0^{-1}(\lambda) = G_\pm^{-1}\begin{pmatrix}1 & \\ & -b_0\end{pmatrix}\mathrm{e}^{-(\frac{z^2}{4}-(\nu+1)\ln z)\sigma_3}\hat{Z}_\pm(\lambda) \qquad \lambda \in \mathcal{P}_\pm \tag{4.106}$$

where

$$\|\hat{Z}_\pm(\lambda) - I\| \leqslant t^{-\delta}C \qquad \forall\lambda \in \mathcal{P}_\pm \qquad \forall s_- \in D(s_-^*; \varepsilon) \qquad t > t_1$$

and

$$\begin{aligned}G_+ &= \begin{pmatrix}\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}(\nu+1)} & \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\mathrm{e}^{\mathrm{i}\pi(\nu+1)}\\ \frac{\sqrt{2\pi}}{\Gamma(\nu+1)}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\nu} & 1\end{pmatrix}\\ G_- &= \begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\nu+1)} & 0\\ 0 & 1\end{pmatrix}.\end{aligned} \tag{4.107}$$

The parameters ν and b_0 are defined in (4.70) and (4.71).

In the matching domains $\mathcal{P}_\pm$ the function $g(\lambda)$ (see (4.88), (4.89)) admits the following, uniform with respect to $s_- \in D(s_-^*; \varepsilon)$ and $t > t_1$, asymptotic representation:

$$\begin{aligned} &2tg(\lambda) = \frac{z^2}{4} - (\nu+1)\ln z - \frac{\mathrm{i}t}{\sqrt{3}} + \frac{\nu+1}{2}\ln t + q + \mathcal{O}(t^{-\frac{1}{2}+3\delta}) \\ &\lambda \in \mathcal{P}_\pm \qquad t \to \infty \end{aligned} \tag{4.108}$$

where

$$q = (\nu+1)\ln 2\sqrt[4]{3} + (\nu+1)\frac{7\mathrm{i}\pi}{12} + \frac{1}{2}\left(\alpha - \beta + \frac{1}{2}\right)\ln\frac{8}{3} + \left(\alpha - \beta + \frac{1}{2}\right)\frac{\mathrm{i}\pi}{3} \tag{4.109}$$

and the variable z and the pure imaginary parameter ν are the same as in (4.69). It should be emphasized that in both equations (4.106) and (4.108) the branch of $\ln z$ is determined by the same rule (4.105).

Simultaneously, from (4.37), (4.38) it follows that

$$\sqrt{\frac{a_+}{\mu}} = 2^{\frac{1}{4}}\sqrt{\frac{2}{3}} + \mathcal{O}\left(\frac{1}{t^\delta}\right) \tag{4.110}$$

as $t \to \infty$ uniformly with respect to $\lambda \in \mathcal{P}_\pm$, $s_- \in D(s_-^*; \varepsilon)$ and $t > t_1$.

Consider the domain $\mathcal{P}_+$ and assume that

$$\lambda, \lambda_0^+ \in \mathcal{P}_+.$$

In virtue of equation (4.108) and the characteristic property (4.74) of the set $\mathcal{S}$, the exponential term in (4.102), i.e. the matrix,

$$\mathrm{e}^{2t\int_{\lambda_0^+}^{\lambda} \Lambda_3 \, \mathrm{d}\lambda \sigma_3}$$

is uniformly bounded and does not affect the power-like error terms in all the other objects involved in equation (4.102). Hence the asymptotic formulae (4.104), (4.106), (4.108), and (4.110) together with the estimates (4.78), (4.60) for the functions $\chi_0(\lambda)$, $\chi_+(\lambda)$ produce the following asymptotic equation for the connection matrix N_+:

$$N_+ = G_+^{-1}\begin{pmatrix} 1 & \\ & -b_0 \end{pmatrix}\mathrm{e}^{(-2tg(\lambda_0^+) - \frac{\mathrm{i}t}{\sqrt{3}} + \frac{\nu+1}{2}\ln t + q)\sigma_3}\{I + n_+(t)\} \tag{4.111}$$

where the matrix function $n_+(t)$ satisfies the uniform estimate,

$$\begin{aligned} &|n_+| < \frac{C}{t^\delta} \qquad \delta = \max_{\tilde{\delta}\in(0,1/6)} \min\{\tilde{\delta}; \tfrac{1}{2} - 3\tilde{\delta}\} = \tfrac{1}{8} \\ &t \geqslant t_0 \qquad s \in D(s_-^*; \epsilon) \qquad \lambda_0^+ \in \mathcal{P}_+. \end{aligned}$$

Similar arguments, based on the restriction,

$$\lambda, \lambda_0^- \in \mathcal{P}_-$$

yield the similar equation for matrix N_-,

$$\begin{aligned} &N_- = G_-^{-1}\begin{pmatrix} 1 & \\ & -b_0 \end{pmatrix}\mathrm{e}^{(-2tg(\lambda_0^-) - \frac{\mathrm{i}t}{\sqrt{3}} + \frac{\nu+1}{2}\ln t + q)\sigma_3}\{I + n_-(t)\} \\ &|n_-| < t^{-\frac{1}{8}}C \\ &t \geqslant t_0 \qquad s \in D(s_-^*; \epsilon) \qquad \lambda_0^- \in \mathcal{P}_-. \end{aligned} \tag{4.112}$$

Step 5. Calculation of the monodromy matrices. The completion of the proof of theorem 4.1. We are now ready to calculate the product of the Stokes matrices indicated in (4.21). In fact, substituting the asymptotic formulae (4.99), (4.100), (4.111), and (4.112) for the matrices $C_\pm$, $N_\pm$ into equation (4.21), we obtain that

$$\begin{aligned}\hat{S}_{-1}\hat{S}_0\hat{S}_1 &= C_-^{-1}N_-^{-1}N_+C_+ = \sigma_1 e^{F_-\sigma_3}\begin{pmatrix}1 & \\ & -b_0^{-1}\end{pmatrix} G_-G_+^{-1}\begin{pmatrix}1 & \\ & b_0\end{pmatrix}\{I+n(t)\}\,e^{F_+\sigma_3}\\ &= \sigma_1 e^{F_-\sigma_3}\begin{pmatrix}1 & -b_0\frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{i\pi(\nu+1)}\\ \frac{i}{b_0}\frac{\sqrt{2\pi}}{\Gamma(\nu+1)} & -e^{2\pi i(\nu+1)}\end{pmatrix}\{I+n(t)\}e^{F_+\sigma_3}\end{aligned} \tag{4.113}$$

where

$$F_- = \frac{it}{\sqrt{3}} - \frac{\nu+1}{2}\ln t - q - \frac{2t}{3} + \frac{\alpha-\beta}{4}\ln 2t - \omega_-$$

$$F_+ = -\frac{it}{\sqrt{3}} + \frac{\nu+1}{2}\ln t + q - \frac{2t}{3} + \frac{\alpha-\beta}{4}\ln 2t + \omega_+$$

and the matrix $n(t)$ satisfies the uniform estimate,

$$|n| < t^{-\frac{1}{8}}C$$
$$t \geqslant t_0 \qquad s \in D(s_-^*;\epsilon).$$

In obtaining (4.113) we also took into account the fact that $\lambda_0^\pm \in \mathcal{P}_\pm$, and hence

$$\sqrt{\frac{\mu}{a_+}}\bigg|_{\lambda_0^-}\sqrt{\frac{a_+}{\mu}}\bigg|_{\lambda_0^+} = 1 + \mathcal{O}\left(\frac{1}{t^\delta}\right),$$

because of (4.110).

Recalling the definitions of the parameters b_0, ν (see (4.70), (4.71)) and $\omega_\pm$, q (see (4.101), (4.109)), equation (4.113) can be rewritten in the folowing form:

$$\begin{aligned}\hat{S}_{-1}\hat{S}_0\hat{S}_1 &= \sigma_1 e^{i\Xi\sigma_3+\Pi\sigma_3}\begin{pmatrix}1 & -\frac{i}{\sqrt[4]{3}a}e^{-i\frac{5\pi}{12}-i\Theta}\frac{\sqrt{2\pi}}{\Gamma(-i\sqrt{3}a^2)}e^{-\pi\sqrt{3}a^2}\\ \frac{i}{\sqrt[4]{3}a}e^{i\frac{5\pi}{12}+i\Theta}\frac{\sqrt{2\pi}}{\Gamma(i\sqrt{3}a^2)} & -e^{-2\pi\sqrt{3}a^2}\end{pmatrix}\\ &\quad\times\{I+n(t)\}e^{-i\Xi\sigma_3+\Pi\sigma_3-\frac{\sqrt{3}}{2}\pi a^2\sigma_3}\end{aligned} \tag{4.114}$$

where

$$\Xi = \frac{t}{\sqrt{3}} - \frac{\sqrt{3}}{2}a^2\ln t - \sqrt{3}a^2\ln 2\sqrt[4]{3} - \frac{\pi}{3}\left(\alpha-\beta-\frac{1}{4}\right) \tag{4.115}$$

$$\Pi = -\frac{2t}{3} + \frac{\alpha-\beta}{4}\ln 2t - \frac{\sqrt{3}}{12}\pi a^2 + \frac{1}{2}\left(\alpha-\beta-\frac{1}{2}\right)\ln\frac{2}{3} \tag{4.116}$$

and the matrix function $n(t)$ satisfies the uniform estimate,

$$|n| < t^{-\frac{1}{8}}C$$
$$t \geqslant t_0 \qquad s_- \in D(s_-^*;\epsilon).$$

The object of our prime interest, i.e. the monodromy parameter $\hat{s}_-$, is given by the 22 entry of the matrix product $\hat{S}_{-1}\hat{S}_0\hat{S}_1$ (see (4.22)). From (4.114) we find that

$$\begin{aligned}&\hat{s}_- = -\frac{i}{\sqrt[4]{3}a}e^{2i\Xi-i\frac{5\pi}{12}-i\Theta}\frac{\sqrt{2\pi}}{\Gamma(-i\sqrt{3}a^2)}e^{-\frac{\sqrt{3}}{2}\pi a^2} + c(t)\\ &|c| < t^{-\frac{1}{8}}C\\ &t \geqslant t_0 \qquad s_- \in D(s_-^*;\epsilon).\end{aligned} \tag{4.117}$$

To complete the proof of theorem 4.1 we only need to notice that, in virtue of the identity,

$$\frac{1}{|\Gamma(\mathrm{i}\tau)|}=-\frac{\mathrm{i}\tau\sin\pi\mathrm{i}\tau}{\pi}\qquad\tau\in\mathbb{R}$$

the equation,

$$-\frac{\mathrm{i}}{\sqrt[4]{3}a}\mathrm{e}^{2\mathrm{i}\Xi-\mathrm{i}\frac{5\pi}{12}-\mathrm{i}\Theta}\frac{\sqrt{2\pi}}{\Gamma(-\mathrm{i}\sqrt{3}a^2)}\mathrm{e}^{-\frac{\sqrt{3}}{2}\pi a^2}=s_- \tag{4.118}$$

is equivalent to the exact formulae (4.9), (4.10) for the functions $a=a(s_-)$ and $\Theta=\Theta(x,s_-)$ suggested in the theorem. One also has to remember that $x=-\sqrt{2t}$.

The proof of theorem 4.1 is completed. □

Remark 4.2. Suppose that instead of exact equations (4.8), the functions $w(x)$ and $w'(x)$ satisfy, as $x\to-\infty$, the asymptotic equations,

$$w=-\frac{2x}{3}+2\sqrt{2}a\cos\Theta+\mathcal{O}\left(\frac{1}{x}\right)\qquad w'=\frac{\mathrm{d}w(x)}{\mathrm{d}x}\qquad\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{O}\left(\frac{1}{x}\right)=\mathcal{O}(1) \tag{4.119}$$

where as before

$$a^2=-\frac{1}{2\sqrt{3}\pi}\ln(1-|s_-|^2)\qquad a>0 \tag{4.120}$$

$$\Theta=\frac{x^2}{\sqrt{3}}-\sqrt{3}a^2\ln(2\sqrt{3}x^2)+\phi \tag{4.121}$$

$$\phi=-\frac{3\pi}{4}-\frac{2\pi}{3}(\alpha-\beta)-\arg\Gamma(-\mathrm{i}\sqrt{3}a^2)-\arg s_-.$$

Then all the estimates we made during the proof of theorem 4.1 would be still valid, provided of course that we are no longer interested in making them uniform with respect to s_-. In particular, we would end up with the equation (cf (4.117)),

$$\hat{s}_-=-\frac{\mathrm{i}}{\sqrt[4]{3}a}\mathrm{e}^{2\mathrm{i}\Xi-\mathrm{i}\frac{5\pi}{12}-\mathrm{i}\Theta}\frac{\sqrt{2\pi}}{\Gamma(-\mathrm{i}\sqrt{3}a^2)}\mathrm{e}^{-\frac{\sqrt{3}}{2}\pi a^2}+\mathcal{O}(t^{-\frac{1}{8}})\equiv s_-+\mathcal{O}(t^{-\frac{1}{8}}). \tag{4.122}$$

Observe now that formulae (4.120) and (4.121) establish a one-to-one correspondence between the real pairs $(\phi,a), a>0, \phi\in\mathbb{R}$ mod 2π and the complex numbers $s_-, 0<|s_-|<1$. Hence we arrive at the following result.

Corollary 4.1. Let α,β,ϕ and a be the real numbers such that $a>0$ and $\alpha-\frac{1}{2}\notin\mathbb{Z}$ (the only restrictions). Suppose that the PIV equation (1.1) has a real solution $w(x)$ satisfying asymptotic condition (4.119) as $x\to-\infty$. Then this solution is unique, i.e.

$$w(x)\equiv w(x;a,\phi)$$

and the corresponding monodromy parameter s_- is given by the equations,

$$|s_-|^2=1-\mathrm{e}^{-2\sqrt{3}\pi a^2} \tag{4.123}$$

and

$$\arg s_-=-\frac{3\pi}{4}-\frac{2\pi}{3}(\alpha-\beta)-\arg\Gamma(-\mathrm{i}\sqrt{3}a^2)-\phi. \tag{4.124}$$

Equations (4.123) and (4.124) are valid without any restrictions on the parameter α.

5. Solution of the inverse monodromy problem: $x \to -\infty$

In this section, we prove the following theorem.

Theorem 5.1. Suppose that the monodromy set (2.20) related to system (4.2) satisfies the conditions,

$$\beta, \alpha \in \mathbb{R} \qquad \alpha - \tfrac{1}{2} \notin \mathbb{Z}$$
$$\overline{s_{-1}s_0} = s_1 s_0 \qquad 0 < |s_-| < 1 \qquad s_- \equiv 1 + s_1 s_0.$$

Then for sufficiently large negative x, the inverse monodromy problem for system (4.2) posed as the problem,

$$s_- \mapsto \{w, w'\} \tag{5.1}$$

is uniquely solvable, and the corresponding solution $w(x)$ of the PIV equation (1.1) is real (for real x) and possesses the following asymptotic behaviour as $x \to -\infty$:

$$w(x) = \hat{w}(x, s_-) + \mathcal{O}((-x)^{-\frac{1}{4}} \ln(-x)). \tag{5.2}$$

In (5.2), $\hat{w}(x, s_-)$ denotes the explicit function introduced in theorem 4.1 by equations (4.8)–(4.10).

Proof of theorem 5.1. Let us first prove the easy part of the statement, i.e. the reality of $w(x)$ and the uniqueness of the solution of the inverse problem (5.1).

The reality of $w(x)$ follows (cf (2.23), (2.24)) from the following lemma.

Lemma 5.1. Let

$$\frac{\mathrm{d}\Psi(\xi)}{\mathrm{d}\xi} = A(\xi)\Psi(\xi) \tag{5.3}$$

be the equation from subclass (4.2) such that for its monodromy data the condition

$$|s_-| \neq 1 \tag{5.4}$$

holds. Assume also that $x < 0$. Then the following three statements are equivalent.

(i) The Stokes multipliers $s_{\pm 1}$, s_0 satisfy the equations,

$$\bar{s}_0 = s_0 \qquad \bar{s}_{-1} = s_1.$$

(ii) The Stokes multipliers $s_{\pm 1}$, s_0 satisfy the equation,

$$\overline{s_{-1}s_0} = s_1 s_0.$$

(iii) The functions $w(x)$, $w'(x)$ are real.

Proof of lemma 5.1. The implication, (i) $\Rightarrow$ (ii), is trivial. The implication (iii) $\Rightarrow$ (i) has been, in fact, proven in section 2 (see (2.21) and (2.22)). Hence, it is enough just to prove the implication,

$$\text{(ii)} \Rightarrow \text{(iii)}.$$

From the equation,

$$\overline{s_{-1}s_0} = s_1 s_0$$

we derive the following representations for the multipliers $s_{\pm 1}$, s_0:

$$s_0 = |s_0|\mathrm{e}^{\mathrm{i}\delta_0} \qquad s_1 = |s_1|\mathrm{e}^{\mathrm{i}\delta_1} \qquad s_{-1} = |s_1|\mathrm{e}^{-\mathrm{i}\delta_1 - 2\mathrm{i}\delta_0} \tag{5.5}$$

($s_1 s_0 \neq 0$, since $|s_-| \neq 1$). This in turn implies that

$$S_0 = \mathrm{e}^{-\mathrm{i}\delta_0\sigma_3}\sigma_3\bar{S}_0^{-1}\sigma_3\mathrm{e}^{\mathrm{i}\delta_0\sigma_3} \qquad \text{and} \qquad S_1 = \mathrm{e}^{-\mathrm{i}\delta_0\sigma_3}\sigma_3\bar{S}_{-1}^{-1}\sigma_3\mathrm{e}^{\mathrm{i}\delta_0\sigma_3}. \tag{5.6}$$

Set

$$\tilde{A}(\xi) = \mathrm{e}^{-\mathrm{i}\delta_0\sigma_3}\sigma_3\bar{A}(\bar{\xi})\sigma_3\mathrm{e}^{\mathrm{i}\delta_0\sigma_3} \tag{5.7}$$

and consider the system

$$\frac{\mathrm{d}\tilde{\Psi}(\xi)}{\mathrm{d}\xi} = \tilde{A}(\xi)\tilde{\Psi}(\xi). \tag{5.8}$$

Denoting $\Psi_k(\xi)$ and $\tilde{\Psi}_k(\xi)$ the canonical solutions corresponding to equations (5.3) and (5.8), respectively, we have that

$$\tilde{\Psi}_{\tilde{k}(k)}(\xi) = \mathrm{e}^{-\mathrm{i}\delta_0\sigma_3}\sigma_3\bar{\Psi}_k(\bar{\xi})\sigma_3\mathrm{e}^{\mathrm{i}\delta_0\sigma_3} \qquad \arg\bar{\xi} = -\arg\xi \qquad k = -1, 0, 1, 2$$
$$\tilde{k}(-1) = 2 \qquad \tilde{k}(0) = 1 \qquad \tilde{k}(1) = 0 \qquad \tilde{k}(2) = -1.$$

This together with (5.6) yield the relations,

$$\tilde{S}_k = S_k \qquad k = -1, 0, 1. \tag{5.9}$$

From (ii) and (5.4) it follows that

$$s_1 + s_{-1} + s_1 s_{-1} s_0 = \frac{|s_-|^2 - 1}{s_0} \neq 0.$$

Therefore the generic condition (2.18) is satisfied, and hence the Stokes matrices $S_{\pm 1}, S_0$ determine uniquely the rest of the monodromy data of equation (5.3). Because of (5.9), the same is true for equation (5.8). Moreover, from (5.9) it follows that both the systems have the same set of the monodromy data. This means that the matrix ratio,

$$F(\xi) \equiv \tilde{\Psi}_0(\xi)\Psi_0^{-1}(\xi)$$

is an entire function which has the asymptotics,

$$F(\xi) \to I \qquad \xi \to \infty$$

in the whole neighbourhood of $\xi = \infty$. Therefore,

$$F(\xi) = I \qquad \forall\xi$$

and we end up with the identity,

$$\tilde{\Psi}_0(\xi) \equiv \Psi_0(\xi)$$

which is followed by the equation,

$$\tilde{A}(\xi) = A(\xi) \qquad \forall\xi \in \mathbb{C}. \tag{5.10}$$

From (5.10) we conclude that the matrix $A(\xi)$ must satisfy the symmetry equation,

$$A(\xi) = \mathrm{e}^{-\mathrm{i}\delta_0\sigma_3}\sigma_3\bar{A}(\bar{\xi})\sigma_3\mathrm{e}^{\mathrm{i}\delta_0\sigma_3} \qquad \forall\xi \tag{5.11}$$

whose ξ^2 term in the 21 component implies that

$$1 = \mathrm{e}^{2\mathrm{i}\delta_0}. \tag{5.12}$$

This equation in view of (5.11) yields the reality of w, w', i.e. the statement (iii). Also, from (5.5) and (5.12) it follows directly that $\bar{s}_0 = s_0$ and $\bar{s}_{-1} = s_1$, i.e. the statement (i). The proof of the lemma is completed. □

The uniqueness of the solution of the inverse monodromy problem for system (4.2) in the setting,

$$s_- \mapsto \{w, w'\}$$

follows from the following proposition

Proposition 5.1. For real $\{w, w'\}$ and under the condition, $|s_-| \neq 1$, the monodromy map for system (4.2), i.e. the map,

$$\{w, w'\} \mapsto s_-$$

is one-to-one.

Proof of proposition 5.1. Given $\alpha, \beta \in \mathbb{R}$, $\alpha - \frac{1}{2} \notin \mathbb{Z}$ consider two systems,

$$\frac{\mathrm{d}\Psi(\xi)}{\mathrm{d}\xi} = A(\xi)\Psi(\xi) \qquad \text{and} \qquad \frac{\mathrm{d}\tilde{\Psi}(\xi)}{\mathrm{d}\xi} = \tilde{A}(\xi)\tilde{\Psi}(\xi)$$

from subclass (4.2) with the real $\{w, w'\}$ and $\{\tilde{w}, \tilde{w}'\}$, respectively, whose monodromy parameters, s_- and $\tilde{s}_-$, coincide,

$$s_- = \tilde{s}_- \qquad |s_-| \neq 1. \tag{5.13}$$

Let $\{S_k, E\}$, $\Psi_k(\xi)$ and $\{\tilde{S}_k, \tilde{E}\}$, $\tilde{\Psi}_k(\xi)$ be the monodromy data and the canonical solutions corresponding to each of the two systems, respectively. Because of equations (5.13), the Stokes matrices $S_{1,0}$ and $\tilde{S}_{1,0}$ are related by the similarity transformation,

$$\tilde{S}_k = \mathrm{e}^{\kappa\sigma_3} S_k \mathrm{e}^{-\kappa\sigma_3} \qquad k = 0, 1 \tag{5.14}$$

with some parameter κ. Due to the reality of $w, w', \tilde{w}$, and $\tilde{w}'$,

$$\bar{s}_0 = s_0 \qquad \bar{\tilde{s}}_0 = \tilde{s}_0 \qquad \bar{s}_1 = s_{-1} \qquad \bar{\tilde{s}}_1 = \tilde{s}_{-1}$$

so that the number $\mathrm{e}^{2\kappa}$ must be real and equation (5.14) must be true for $k = -1$ as well. Arguing as in the proof of lemma 5.1, we conclude that the two systems with the coefficient matrices,

$$\tilde{A}(\xi) \qquad \text{and} \qquad \mathrm{e}^{\kappa\sigma_3} A(\xi) \mathrm{e}^{-\kappa\sigma_3}$$

respectively, have the same set of the monodromy data, and hence the relation,

$$\tilde{A}(\xi) = \mathrm{e}^{\kappa\sigma_3} A(\xi) \mathrm{e}^{-\kappa\sigma_3} \qquad \forall\xi$$

is present. Considering again the ξ^2 term in the 21 component of the last equation, we obtain that

$$\mathrm{e}^{-2\kappa} = 1$$

and therefore,

$$\tilde{A}(\xi) = A(\xi) \qquad \forall\xi$$

or

$$\{w, w'\} = \{\tilde{w}, \tilde{w}'\}$$

which completes the proof of proposition 5.1. □

To prove the most interesting part of the statement, i.e. the existence of the solution of the inverse monodromy problem (5.1) and the asymptotics (5.2), we shall use theorem 4.1 and Kitaev's method [11].

Let s_-^* be the complex number satisfying the inequality,

$$0 < |s_-^*| < 1$$

and let $D(s_-^*; \varepsilon)$ be the closed disk,

$$D(s_-^*; \varepsilon) = \{s_- \in \mathbb{C} : |s_- - s_-^*| \leqslant \varepsilon\} \qquad 0 < \varepsilon < \min\{1 - |s_-^*|, |s_-^*|\} \tag{5.15}$$

as in theorem 2 (cf (4.7)). Taking $s_- \in D(s_-^*; \varepsilon)$, we consider the coefficient functions $\hat{w}(x, s_-)$, $\hat{w}'(x, s_-)$, and the corresponding monodromy data $\hat{s}_-(x, s_-)$. From the general theory of systems of ODEs with rational coefficients (see e.g. [16]) it follows that the canonical solutions Ψ_k of system (4.2) are smooth functions of x, $\hat{w}$, $\hat{w}'$. This implies that $\hat{s}_-(x, s_-)$ is a continuous function on $(-\infty; -1] \times D(s_-^*; \varepsilon)$.

Let us introduce the function $g(x, s_-)$ by

$$\hat{s}_-(x, s_-) = s_- + g(x, s_-). \tag{5.16}$$

The continuity of the function $\hat{s}_-$ and theorem 4.1 imply that:

(1) the function $g(x, s_-)$ is continuous on $(-\infty; -1] \times D(s_-^*; \varepsilon)$,

(2) there exist the constants $C = C(s_-^*; \varepsilon) > 0$ and $x_0 = x_0(s_-^*; \varepsilon) < -1$ such that

$$|g(x, s_-)| \leqslant (-x)^{-\frac{1}{4}} C \qquad \forall x < x_0 \qquad \forall s_- \in D(s_-^*; \varepsilon). \tag{5.17}$$

Now let us consider the equation,

$$s_- + g(x, s_-) = s_-^* \qquad s_- \in D(s_-^*; \varepsilon). \tag{5.18}$$

Introducing the variable $\tau = s_-^* - s_-$ and the function $\tilde{g}(x, \tau) \equiv g(x, s_-^* - \tau)$, one can rewrite equation (5.18) as

$$\tilde{g}(x, \tau) = \tau \qquad \tau \in D(0; \varepsilon) \equiv \{\tau : |\tau| \leqslant \varepsilon\}. \tag{5.19}$$

Picking any $x < x_1$, $x_1 = -\max\{(-x_0); (C/\varepsilon)^4\}$, we conclude that the function $\tilde{g}(x, \cdot)$ is a *continuous function from the compact disk* $D(0; \varepsilon)$ *into itself*. Thus the Brouwer fixed point theorem implies that for each $x \leqslant x_1$, there exists at least one solution of the equation (5.18).

One can see that the solution of (5.18) is unique for $x < \min\{x_1; x_2\}$, where

$$x_2 = -3\sqrt{2}\left(-\frac{1}{2\sqrt{3}\pi}\ln(1 - (|s_-^*| + \varepsilon)^2)\right)^{1/2} < 0.$$

Indeed, let $x < x_1$ and suppose that there are two complex numbers s_-, $\tilde{s}_-$ satisfying (5.18). Consider the pairs $\{\hat{w}(x, s_-), \hat{w}'(x, s_-)\}$ and $\{\hat{w}(x, \tilde{s}_-), \hat{w}'(x, \tilde{s}_-)\}$. For the corresponding monodromy data $\hat{s}_-(x, s_-)$ and $\hat{s}_-(x, \tilde{s}_-)$ we have

$$\hat{s}_-(x, s_-) = s_- + g(x, s_-) = s_-^* \qquad \hat{s}_-(x, \tilde{s}_-) = \tilde{s}_- + g(x, \tilde{s}_-) = s_-^*$$

so that $\hat{s}_-(x, s_-) = \hat{s}_-(x, \tilde{s}_-)$. Due to proposition 5.1, we obtain that

$$\hat{w}(x, s_-) = \hat{w}(x, \tilde{s}_-) \qquad \hat{w}'(x, s_-) = \hat{w}'(x, \tilde{s}_-)$$

or

$$a \cos\Theta(x) = \tilde{a} \cos\tilde{\Theta}(x)$$
$$a\left(1 - \frac{3a^2}{x^2}\right)\sin\Theta(x) = \tilde{a}\left(1 - \frac{3\tilde{a}^2}{x^2}\right)\sin\tilde{\Theta}(x).$$

In terms of the new complex variables $Z = a\mathrm{e}^{\mathrm{i}\Theta}$, $\tilde{Z} = \tilde{a}\mathrm{e}^{\mathrm{i}\tilde{\Theta}}$, these equations become

$$Z - \frac{3\mathrm{i}}{x^2}|Z|^2 \mathrm{Im}\, Z = \tilde{Z} - \frac{3\mathrm{i}}{x^2}|\tilde{Z}|^2 \mathrm{Im}\, \tilde{Z}.$$

This yields the following inequality:

$$|Z-\tilde{Z}| \leqslant \frac{9C_0}{x^2}|Z-\tilde{Z}|$$

where C_0 is defined by

$$C_0 = \max_{s_- \in D(s_-^*;\varepsilon)} a^2 \leqslant -\frac{1}{2\sqrt{3}\pi}\ln(1-(|s_-^*|+\varepsilon)^2) \equiv \frac{1}{18}x_2^2.$$

If $x \leqslant x_2 < 0$, we conclude that $Z = \tilde{Z}$, i.e. $a = \tilde{a}$ and $\Theta = \tilde{\Theta}$ (mod 2π), and therefore $s_- = \tilde{s}_-$.

Assuming that

$$x < x_3 \equiv \min\{x_1; x_2\}$$

we will denote the unique solution of (5.18) as $s_-(x; s_-^*)$. This function:

(a) is defined for $x < x_3$;

(b) for all $x < x_3$ satisfies the equation

$$s_-(x, s_-^*) + g(x, s_-(x, s_-^*)) = s_-^* \qquad s_-(x, s_-^*) \in D(s_-^*; \varepsilon)$$

(c) for all $x < x_3$ satisfies the inequality

$$|s_-(x, s_-^*) - s_-^*| \leqslant x^{-\frac{1}{4}} C(s_-^*; \varepsilon) < \varepsilon.$$

Now, the last step of the proof. Let us define

$$\tilde{w}(x, s_-^*) = \hat{w}(x, s_-(x, s_-^*)) \qquad \tilde{w}'(x, s_-^*) = \frac{\partial}{\partial x}\hat{w}(x, s_-)|_{s_-=s_-(x,s_-^*)} \qquad x < x_3.$$

Taking these functions as the coefficients in the system (4.2), we find that the corresponding monodromy data satisfies the equation

$$\tilde{s}_- = \hat{s}_-(x, s_-(x, s_-^*)) = s(x, s_-^*) + g(x, s_-(x, s_-^*)) \equiv s_-^*$$

for all $x < x_3$ and hence does not depend on x. This means that:

(i) for any $x < x_3$, the pair $\{\tilde{w}(x, s_-^*), \tilde{w}'(x, s_-^*)\}$ is a solution of the inverse monodromy problem (5.1) corresponding to the monodromy data s_-^*;

(ii) the function $\tilde{w}(x, s_-^*)$ coincides with the solution $w(x, s_-^*)$ of the PIV equation (1.1) corresponding to the monodromy parameter s_-^*.

This implies the equation

$$\begin{aligned} &w(x, s_-^*) = \hat{w}(x, s_-(x, s_-^*)) \qquad x < x_3(s_-^*; \varepsilon) \\ &\forall s_-^* : 0 < |s_-^*| < 1 \qquad \text{and} \qquad 0 < \varepsilon < \min\{1-|s_-^*|, |s_-^*|\} \end{aligned} \tag{5.20}$$

which completes the proof of theorem 5.1. In fact, it remains to use estimate (c) for function $s_-(x, s_-^*)$ and the smoothness of the functions $a = a(\operatorname{Re} s_-, \operatorname{Im} s_-)$, $\phi = \phi(\operatorname{Re} s_-, \operatorname{Im} s_-)$ for $s_- \in D(s_-^*; \varepsilon)$. □

Remark 5.1. It has already been noticed (see remark 4.1) that the map,

$$0 < |s_-| < 1 \qquad s_- \mapsto (\phi, a) \qquad a > 0 \qquad \phi \in \mathbb{R} \bmod 2\pi$$

given by the equations,

$$a^2 = -\frac{1}{2\sqrt{3}\pi}\ln\left(1-|s_-|^2\right) \qquad a > 0 \tag{5.21}$$

$$\phi = -\frac{3\pi}{4} - \frac{2\pi}{3}(\alpha-\beta) - \arg\Gamma(-\mathrm{i}\sqrt{3}a^2) - \arg s_- \tag{5.22}$$

is a bijection. This fact, theorem 5.1, and corollary 4.1 imply the local asymptotic result formulated in theorem 1.2.

6. Proof of theorem 1.1. Connection formulae.

Let the parameters α, β, and k^2 satisfy the conditions of theorem 1.1, and let $w(x; k^2)$ denote the Clarkson–McLeod solution of the PIV equation (1.1) characterized by the boundary condition (1.3) as $x \to +\infty$. Assume also that in theorem 3.1 the monodromy data are chosen so that the equations,

$$\bar{s}_0 = s_0 \qquad \bar{s}_1 = s_{-1}$$
$$s_1 s_0 = -\frac{2(2\pi)^{3/2}\mathrm{e}^{-\mathrm{i}\pi\alpha}}{\Gamma(\frac{1}{2}-\alpha)}k^2$$

take place. We note that this, in particular, implies that

$$0 \neq s_1 s_0 \mathrm{e}^{\mathrm{i}\pi\alpha} \in \mathbb{R}$$

and hence (cf (2.26), (2.27)),

$$s_2 = s_1 + s_3 = 0.$$

Taking into account the uniqueness of the solution $w(x; k^2)$ and comparing the asymptotics (1.3) and (3.1), we conclude that the Painlevé transcendent $w(x; k^2)$ is real for real x and the direct monodromy map,

$$w(x; k^2) \mapsto s_-(k^2)$$

is given by the explicit formula,

$$s_-(k^2) = 1 - \frac{2(2\pi)^{3/2}\mathrm{e}^{-\mathrm{i}\pi\alpha}}{\Gamma(\frac{1}{2}-\alpha)}k^2. \tag{6.1}$$

To complete the proof of theorem 1.1 we only need to refer to theorem 5.1 noticing that inequality (1.9) is equivalent to the inequality,

$$0 < |s_-| < 1$$

if s_- is given by (6.1).

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